Dynamic Capital Structure
and
Stochastic Interest Rates

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Abstract

This paper investigates the effect of the term structure of interest rates on the corporate optimal total capital structure. Having derived closed form solutions for the values of debt, tax benefits, bankruptcy costs and recapitalization costs, we numerically solve for the optimal total capital structure: leverage, debt service, maturity, call provisions and priority.

The value of corporate debt is not only interlinked with capital structure but also related to the dynamics of interest rates. Corporate debt is not only a contingent claim on the underlying firm's assets, but also an interest-rate derivative as well. Though joint knowledge of debt's value and capital structure is a necessary condition to determine both corporate debt value and optimal capital structure, one cannot solve sufficiently accurately for the value of debt, and hence optimal capital structure, without knowing the dynamics of interest rates.

This article models the capital structure choice in a continuous-time framework, considering the effect of the dynamics of both underlying technologies and interest rates. We first derive the fundamental valuation equation for the corporate debt, tax benefits, bankruptcy costs and recapitalization costs in the context of a dynamic recapitalization problem. Having solved analytically for the net benefit of debt as a function of firm's cash flow and interest rates, we then solved numerically for the value maximizing optimal capital structure: debt coupon, debt principle, debt maturity and call policy. The resulting optimal dynamic capital structure policy depends on refinancing cost, bankruptcy costs, tax advantages, the underlying assets' dynamics, as well as the term structure of interest rates. This, in turn, allows us to address number of interesting questions. For instance, how do the driving factors of interest rate dynamics, i.e. short rates, long rates, and volatility, influence optimal debt ratio? Does parametric structure of the interest rate dynamics (i.e. shape of the yield curve) affect the dynamic financing decisions of the firm? To what extent does the term structure of interest rate have any impact on maturity choice and how? And lastly, does priority structure of debt relate to term structure of interest rates, and if yes how?

Though two strands of the literature, corporate finance[1] and asset pricing[2] have put forth number of studies regarding corporate debt valuation, our understanding of the effect of term structure of interest rates on the capital structure policy and hence, corporate debt pricing remains limited. In one hand, the extant literature on capital structure attempts to jointly value corporate claims and determine the value-maximizing mix of corporate securities. However, the existing models assume a flat term structure of interest rates. While on the other hand, the studies on fixed income securities pricing thrive to present a complete pricing model for risky debt as a contingent claim on the term structure of interest rates. Nonetheless, these models assume capital structure to be fixed. Despite indications from both camps[3] on the importance of the interest rate dynamics in determination of the optimal capital structure and value of corporate debt, we know little as to how structural characteristic of the dynamics of interest rates affect the capital structure choice.

In our model, a firm is endowed with a technology that determines the firm's cash flow dynamics. To finance its operations, the firm may choose to issue debt or equity and can alter its capital-maturity-priority structure at any time. In this setting, the optimal dynamic financing policy (i.e. amount, periodic service, maturity, and priority of the debt as well as call provisions) is characterized by a tradeoff between the tax advantages of debt and bankruptcy and recapitalization costs in presence of stochastic interest rates. By varying driving factors of the term structure as well as parametric structure of interest rate dynamics, we explore the impact of the term structure of interest rates on tax advantages of debt as well as costs of debt financing particularly the probability

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[1] Fischer, Heinkel, Zechner (1989), Leland (1994, and 1998), Leland and Toft (1996), and MallaBerra (1999) are among the handful of paper who developed continuous time models of capital structure.

[2] Merton (1974), Black and Cox (1976), Brennan and Schwartz (1978), Kim, Ramaswamy, Sundaresan (1993), and Lonsgtaff and Schwartz (1995) are among a handful of studies that presented models of corporate debt pricing.

[3] Fischer, Heinkel and Zechner (1989) found that initial leverage ratio and optimal range of dent ratio is decreasing in risk-free rate. They posited that this is due mainly to increased tax advantages of debt when risk-free increases. Kim, Ramaswamy, Sundaresan (1993), Leland (1994, and 1998), and Longstaff and Schwartz (1995) found that optimal debt ratio is increasing in risk-free rate. By in large across all these models, the probability of bankruptcy is decreasing in risk-free rate, hence leading to less debt financing costs (bankruptcy costs).

of bankruptcy. This, in turn, enables us to examine the influence of the term structure of interest rates on the firm's optimal leverage, maturity and priority choice.

Our paper extend the existing literature in two important ways. First, we extend the existing models of dynamic capital structure to account for stochastic nature of interest rates. This, in turn, enables us to study the impact of the behavior of interest rates on optimal capital structure; leverage, maturity, call provisions, bankruptcy procedures, and priority. We also provide closed–form solution for the value of a perpetual debt under stochastic interest rates, inflation and jumps.

This paper is organized as follows. Section I presents an optimal stochastic control model in which firm selects optimal dynamic financing decision in presence stochastic interest rates. Section II examines the impact of the term structure of interest rates on the optimal leverage and maturity choice using numerical procedures. Section III discusses the robustness of the results under alternative model assumptions and parameter values, and lastly, section VI concludes.

I. Model

Following the traditional studies of capital structure, we assume that investment decisions are fixed and exogenous of the financing policy. Henceforth, we model the value of a levered firm as a function of cash flows generated by the firm, term structure of interest rates, leverage ratio recapitalization costs, and maturity.

In the spirit of extant literature[4] and general equilibrium asset pricing model of Cox, Ingersoll and Ross (1985) (hereafter CIR), we employ following assumptions:

(A0) The firm's investment decision in essence determines its technology and hence the dynamics of its cash flows. However, we assume that the firm's financing decision is conditioned on the existing technology, thus there are no interactions between capital budgeting and financing decisions.

(A1) Under technical conditions, we assume that there exists a vector of $\boldsymbol{R}(t)$ state variables that governs the dynamics of interest rates[5].

(A2) There exists a differential tax system: Corporations are taxed instantaneously at constant rate of τ_c. There is no loss-offset provision of taxes. Individuals, however, pays taxes on interest and equity income at rate τ_p, but they are not taxed on their capital gain income.

(A3) Investors, individual and corporations, can trade at fair prices within the boundaries of transaction costs.

(A4) The firm's capital structure consists of debt and equity. The firm can issue coupon debt with coupon ι, and par value, B, and average maturity, M. The firm can also

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[4] Merton (1974), Black and Cox (1976), Brennan and Schwartz (1984), Fishcer, Heinkel, and Zechner (1989), Leland (1994, 1998), , Mauer and Triantis (1994), Leland and Toft (1996), and MellaBarra (1999). These studies derived continuous time valuation models of firm's value. In essence, these models have relaxed one or more assumptions of Modigliani and Miller (1958) by incorporating bankruptcy, taxes, and transaction costs. Leland (1998) in fact takes an step further to explore the effect of agency costs on optimal capital structure particularly on maturity choice. Mauer and Triantis explore the interaction between investment decisions and financing policy. In spirit of these studies, we incorporate bankruptcy costs, differential tax treatment, and transaction costs as major value deriving frictions in our model. Leland (1994) indeed finds that the impact of agency costs on capital structure is negligible. For tractability, we do however, assume that investment policies are fixed.

[5] Dai and Singleton (1999) show that under certain technical conditions, any AY representation of term structure of interest rate can be transformed to Ar representation. In our case, $\boldsymbol{R}(t)$ is the driving vector of state variables that governs the term structure of interest rates in a Ar fashion.

adjust its capital structure at any time, but must pay recapitalization costs by doing so. There are fixed recapitalization costs, κ_F, as well as proportional costs, κ_V, on absolute adjustments of debt's face value adjustments, $|\Delta B|$. Hence, the total recapitalization cost is of the form $\kappa_F + \kappa_V |\Delta B|$.

(A5) Following Leland (1994, and 1998), we assume that debt is continuously retired at par at a constant rate of m. This in fact resembles to a sinking fund that repurchases debt at par. Though no explicit maturity is stated for the debt, Leland shows that average maturity is equal to $1/m$, meaning as the debt repurchase rate increase, the maturity debt shortens. Nonetheless, we assume that this continuous retirement and reissuance of debt is costly. We assume this cost is proportional at a rate κ_m to retirement rate, mB.

(A6) The bond indenture provisions prohibits stockholders from selling assets to pay any dividends, and maintains absolute priority for bondholders. In occasion of default, the firm incurs a total deadweight cost is equal to $g\, v(C^L,\boldsymbol{R}^L)$, where $0 \leq g < 1$, and $v(C^L,\boldsymbol{R}^L)$ is the unlevered value of the firm at the time of default. We maintain that smooth pasting conditions are to be met, in other words, at default point $(C^L,\boldsymbol{R}^L)$, $\partial v(C,\boldsymbol{R}) / \partial C \equiv 0$, and $\partial v(C,\boldsymbol{R}) / \partial \boldsymbol{R} \equiv 0$. Note that the bond indenture demands that the firm to make the continuous debt services, interest and retirement of par, before paying any dividend to stockholders, hence, $C^L \geq \iota + mB$.

(A7) Similar to previous studies [Goldstein et al (1997) and Leland (1998)], recapitalization occurs when bonds are called. One, if the value of corporate bond is equal to some pre-specified call price, $B^C = (1+\beta)B$, where β is known and fixed. This is indeed departs from reality since most call prices decrease in time. In such event, firm can recall the outstanding bonds and issue a new bond with new face value. For simplicity, we assume that at recapitalization, the optimal capital structure, $\boldsymbol{X} = (B, \iota, C^U, \boldsymbol{R}^U, C^U, \boldsymbol{R}^L, m)$, is adjusted with a factor of proportionality $\rho^U \equiv v(C^U,\boldsymbol{R}^U)/v(C_0,\boldsymbol{R}_0)$, where v is the unlevered value of the firm.

A. The Valuation of Firm

In this section we derive the fundamental valuation equations for the value of debt, tax benefits, bankruptcy costs, and recapitalization costs for a general recapitalization policy. Note that the value of debt is merely sum of the unlevered value and tax benefits less of bankruptcy and recapitalization costs. Then, we define the boundary and smooth-pasting conditions for each equation. Since the resulting valuation equation is

Let's define cash flow generated by the firm's technology, C, for which the progression through time is govern by a geometric Brownian motion such as follow:

$$dC = \alpha(C,t)\ dt + \sigma(C,t)\ dW \tag{1}$$

where, C is cash flows generated by the firm, $\alpha(C,t)$ is the drift of the cash flow process, $\sigma(C,t)$ is the instantaneous volatility of the cash flow process, and W is a standard Wiener process. Though drift and volatility can obviously be functions of time and cash flows, for simplicity and without loss of generality, we assume that both these variables are constant.

We also assume that the term structure of interest rate are govern by a vector of state variable, $\boldsymbol{R}(t)$, and the dynamics of interest is as follow:

$$d\boldsymbol{R} = \boldsymbol{\mu}(\boldsymbol{R},t)\ dt + \boldsymbol{S}(\boldsymbol{R},t) d\boldsymbol{W}^{R} \tag{2}$$

where, $\boldsymbol{\mu}(\boldsymbol{R},t)$ is the drift vector of the interest rate dynamics, $\boldsymbol{S'S}$ the non-negative positive covariance matrix of the term structure process, and $\boldsymbol{W}^R$ ($\boldsymbol{W}^R \perp W$) is a vector of orthogonal standard Wiener processes.

Using aforementioned assumptions and adapting CIR fundamental valuation equation, we can write the valuation equation for the debt and equity as following:

$$\tfrac{1}{2}\sigma^2 D_{CC} + \hat{\alpha} D_C + \tfrac{1}{2} tr(\boldsymbol{S'}\,\boldsymbol{S}) D_{\boldsymbol{rr}} + \hat{\boldsymbol{\mu}} D_{\boldsymbol{r}} - (r+m)(1-\tau_p)D + (1-\tau_p)(\iota + mB) = 0 \tag{3}$$

where, r is the instantaneous rate, $\hat{\alpha}$ and $\hat{\boldsymbol{\mu}}$ are risk-adjusted returns due to cash flows and underlying term structure, respectively. Assuming that risk premium on factors are proportional to variance of the underlying process (like CIR), then the drift parameters can be defined as functions of initial drifts, risk premia, and dividend yield.

We also know that the firms levered value is given by:

$$V(C,\boldsymbol{R};\boldsymbol{X}) = D(C,\boldsymbol{R};\boldsymbol{X}) + E(C,\boldsymbol{R};\boldsymbol{X}) = v(C,\boldsymbol{R}) + TB(C,\boldsymbol{R};\boldsymbol{X}) - KC(C,\boldsymbol{R};\boldsymbol{X}) - BC(C,\boldsymbol{R};\boldsymbol{X}) \tag{4}$$

where v is the unlevered value of the firm, $TB(C,\boldsymbol{R};\boldsymbol{X})$ tax benefits, $KC(C,\boldsymbol{R};\ \boldsymbol{X})$ is the total recapitalization costs, and $BC(C,\boldsymbol{R};\ \boldsymbol{X})$ is the bankruptcy cost. By local expectation, the unlevered value of the firm is essentially equal to:

$$v(C,\boldsymbol{R}) = \boldsymbol{E}_Q\left[\int_0^\infty C(u)\ \exp\left\{\int_0^\infty r(s)ds\right\}\ du\right] \tag{5}$$

Using CIR pricing technique and Leland (1998) valuation scheme, for any contingent claim on the firms cash flows we can write a fundamental valuation equation follows:

$$\tfrac{1}{2}\sigma^2 F_{CC} + \hat{\alpha} F_C + \tfrac{1}{2} tr(\boldsymbol{S'}\,\boldsymbol{S}) F_{\boldsymbol{rr}} + \hat{\boldsymbol{\mu}} F_{\boldsymbol{r}} - r(1-\tau_p)F + CF(C,\boldsymbol{R};B) = 0 \tag{6}$$

where $CF(C,\boldsymbol{R};\ \boldsymbol{X})$ is the after-tax continuous cash flow generated by the contingent claim. In case of tax benefits, TB, for $C^L \leq C \leq C^U$ and $\boldsymbol{R}^L \leq \boldsymbol{R} \leq \boldsymbol{R}^U$, the above equation is written as

$$\tfrac{1}{2}\sigma^2 TB_{CC} + \hat{\alpha} TB_C + \tfrac{1}{2} tr(\boldsymbol{S'}\,\boldsymbol{S}) TB_{\boldsymbol{rr}} + \hat{\boldsymbol{\mu}} TB_{\boldsymbol{r}} - r(1-\tau_p)TB + (\tau_c - \tau_p)\iota = 0 \tag{7}$$

The total recapitalization costs, $KC = KC^* + \kappa_F + \kappa_V B$, where KC^* is the recapitalization cost due to continuous retirement. For $C^L \leq C \leq C^U$ and $\boldsymbol{R}^L \leq \boldsymbol{R} \leq \boldsymbol{R}^U$, the fundamental valuation equation of KC^* can be written as

$$\tfrac{1}{2}\sigma^2 KC^*_{CC} + \hat{\alpha} KC^*_C + \tfrac{1}{2} tr(\boldsymbol{S'}\,\boldsymbol{S}) KC^*_{\boldsymbol{rr}} + \hat{\boldsymbol{\mu}} KC^*_{\boldsymbol{r}} - r(1-\tau_p)KC^* + (1-\tau_p)\kappa_m mB = 0 \tag{8}$$

And, lastly for the bankruptcy costs, BC, for $C^L \leq C \leq C^U$ and $\boldsymbol{R}^L \leq \boldsymbol{R} \leq \boldsymbol{R}^U$, the fundamental valuation is defined as

$$\tfrac{1}{2}\sigma^2\, BC_{CC} + \hat{\alpha}\, BC_C + \tfrac{1}{2} tr\left(\boldsymbol{S}'\,\boldsymbol{S}\right) BC_{\boldsymbol{rr}} + \hat{\boldsymbol{\mu}}\, BC_{\boldsymbol{r}} - r\left(1-\tau_p\right) BC = 0 \tag{9}$$

Since the objective of the firm is to maximize the total levered value, or equivalently to maximize the net benefits of debt (i.e. tax benefits minus all costs). Thus, we can write the objective function as

$$\begin{aligned} &\max_{\boldsymbol{X}|\Phi} \left\{ NB(C,\boldsymbol{R};\boldsymbol{X}) \equiv NB^*(C,\boldsymbol{R};\boldsymbol{X}) - \kappa_F - \kappa_V\, B \equiv TB(C,\boldsymbol{R};\boldsymbol{X}) - KC(C,\boldsymbol{R};\boldsymbol{X}) - BC(C,\boldsymbol{R};\boldsymbol{X}) \right\} = \\ &\max_{\boldsymbol{X}|\Phi} \Big\{ NB(C,\boldsymbol{R};\boldsymbol{X}) \equiv NB^*(C,\boldsymbol{R};\boldsymbol{X}) - \kappa_F - \kappa_V\, B \;\Big|\; \tfrac{1}{2}\sigma^2\, NB^*_{CC} + \hat{\alpha}\, C\, NB^*_C + \\ &\qquad + \tfrac{1}{2} tr\left(\boldsymbol{S}'\,\boldsymbol{S}\right) NB^*_{\boldsymbol{rr}} + \hat{\boldsymbol{\mu}} V_{\boldsymbol{r}} - r\left(1-\tau_p\right) NB^* + \left(\tau_c - \tau_p\right)\iota - \left(1-\tau_p\right)\kappa_m mB - \kappa_F - \kappa_V B = 0 \Big\} \end{aligned} \tag{10}$$

where, $NB(C,\boldsymbol{R};\boldsymbol{X}) \equiv TB(C,\boldsymbol{R};\boldsymbol{X}) - KC(C,\boldsymbol{R};\boldsymbol{X}) - BC(C,\boldsymbol{R};\boldsymbol{X})$, $\boldsymbol{\Phi} = (\phi^C, \boldsymbol{\phi}^{\boldsymbol{R}})$ is the market-clearing vector of the factors' risk premia for corporate debt, and $C^L \leq C \leq C^U$ and $\boldsymbol{R}^L \leq \boldsymbol{R} \leq \boldsymbol{R}^U$.

B. Boundary Conditions:

For equation (3), we know that at the lower bounds, when bankruptcy occurs, the firm cash flow is at C^L and term structure of interest rates is at $\boldsymbol{R}^L$. In occasion of default the dead–weight bankruptcy cost is proportional to the unlevered value of the firm, $v(C^L,\boldsymbol{R}^L)$, and hence, the binding lower boundary for the value of corporate debt can be rewritten as:

$$D(C^L, \boldsymbol{R}^L; \boldsymbol{X}) = \left(1-g\right) v(C^L, \boldsymbol{R}^L) \tag{11}$$

Note that at default, the limited liability provision demands that the value of equity to be at its maximal, hence, we require following smooth–pasting conditions to be held:

$$\lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L, \boldsymbol{R}^L; \boldsymbol{X}) \big/ \partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L, \boldsymbol{R}^L; \boldsymbol{X}) \big/ \partial \boldsymbol{R} = 0$$

Where, $\{C^L, \boldsymbol{R}^L\}$is the bankruptcy triggering duply, and $E(C,\boldsymbol{R};\boldsymbol{X})$ is the value of equity.

The equation (3) is also subjected to call provisions which govern the upper boundary condition. Any bond issue can be called back at a pre–specified call price, proportional to the bond's face value. Hence, the binding upper boundary for the value of corporate debt is:

$$D(C^U, \boldsymbol{R}^U; \boldsymbol{X}) = B^C = \left(1+\beta\right) B \tag{12}$$

where $\{C^U, \boldsymbol{R}^U\}$ is the triggering duple for recapitalization. Since this duple represents a plane, at any point on the plane the following smooth–pasting conditions has to hold:

$$\lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U, \boldsymbol{R}^U; \boldsymbol{X}) \big/ \partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U, \boldsymbol{R}^U; \boldsymbol{X}) \big/ \partial \boldsymbol{R} = 0 \tag{13}$$

For equation (7), we have that $TB(C^U,\boldsymbol{R}^U;\boldsymbol{X}) = \rho^U\, TB(C_0,\boldsymbol{R}_0;\boldsymbol{X})$, and $\rho^U \equiv v(C^U,\boldsymbol{R}^U)/v(C_0,\boldsymbol{R}_0)$. Also, since we assumed that there are no loss-offset provisions, thus $TB(C^L,\boldsymbol{R}^L;\boldsymbol{X}) = 0$.

For equation (8), we have that $KC^*(C^U,\boldsymbol{R}^U;\boldsymbol{X}) = \rho^U\,[KC^*(C_0,\boldsymbol{R}_0;\boldsymbol{X}) + \kappa_V \mathrm{B}] + \kappa_F$, and $\rho^U \equiv v(C^U,\boldsymbol{R}^U)/v(C_0,\boldsymbol{R}_0)$. Also, since we assumed that there are no loss-offset provisions, thus $KC^*(C^L,\boldsymbol{R}^L;\boldsymbol{X}) = 0$

For equation (7), we have that $BC(C^U,\boldsymbol{R}^U;\boldsymbol{X}) = \rho^U\,BC(C_0,\boldsymbol{R}_0;\boldsymbol{X})$, and $\rho^U \equiv v(C^U,\boldsymbol{R}^U)/v(C_0,\boldsymbol{R}_0)$. Also, since we assumed that there are dead-weight costs of bankruptcy, thus $BC(C^L,\boldsymbol{R}^L;\boldsymbol{X}) = g\,v(C^L,\boldsymbol{R}^L)$.

Since net benefits of debt, $NB(C,\boldsymbol{R};\boldsymbol{X}) \equiv TB(C,\boldsymbol{R};\boldsymbol{X}) - KC(C,\boldsymbol{R};\boldsymbol{X}) - BC(C,\boldsymbol{R};\boldsymbol{X})$, the boundary-initial conditions are $NB(C^U,\boldsymbol{R}^U;\boldsymbol{X}) = \rho^U\,NB(C_0,\boldsymbol{R}_0;\boldsymbol{X}) - \rho^U\,\kappa_V\,B - \kappa_F$, and $\rho^U \equiv v(C^U,\boldsymbol{R}^U)/v(C_0,\boldsymbol{R}_0)$, and also $NB(C^L,\boldsymbol{R}^L;\boldsymbol{X}) = -\,g\,v(C^L,\boldsymbol{R}^L)$.

II. Optimal Capital-Maturity Structure and Risk Free Rate

To analyze the impact of interest rates and parametric structure of the its dynamics on the optimal capital-maturity choice, we assume that term structure of interest rates can be appropriately modeled by a one-factor CIR model. As discussed earlier, the optimal decision for a firm would be to choose the set of control variables, namely optimal (i.e. value-maximizing) capital-maturity structure, $\boldsymbol{X}$, given the market-clearing vector of risk premia, $\boldsymbol{\Phi}$, such that equation (10) is satisfied.

To solve equation (10), one essentially ought to solve a free-boundary problem. However, given the orthogonality of cash flow and interest rate processes, one can solve analytically for the objective function in equation (10). Though we cannot analytically solve for the solution to maximization problem, the numerical procedure can be easily implemented to find the optimal capital-maturity structure.

Proposition 1. *Assume an economy in which the term structure of interest rates follows a CIR dynamics (Feller process) such as following:*

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r \sqrt{r(t)}\,dW^r \tag{14}$$

and the evolutions of the firm's cash flow is governed by an Ornstein-Uhlenbech process such as following:

$$dC(t) = \alpha\,C(t)\,dt + \sigma\,dW \tag{15}$$

Given that the previous processes are orthogonal, then the value of a contingent claim, $F(C,r;\boldsymbol{X})$, on the firm's cash flows, C, and term structure of interest rates which perpetually pays a known amount, δ, can be expressed as:

$$\tfrac{1}{2}\sigma^2 F_{CC} + \left(\alpha C - \lambda_C \sigma^2\right)F_C + \tfrac{1}{2}\sigma_r^2 r F_{rr} + \left[\kappa\theta - \left(\kappa + \lambda_r \sigma_r^2\right)r\right]F_r - r\left(1 - \tau_p\right)F + \delta(t) = 0 \tag{16}$$

where, λ_r is risk premium parameter of the contingent claim $F(C,r;\boldsymbol{X})$ with respect to interest rates, λ_C is risk premium parameter of the contingent claim $F(C,r;\boldsymbol{X})$ with respect to cash flows. For which there exist a general solution, $F(C,r;\boldsymbol{X})$ of the form:

$$F(C,r;\boldsymbol{X})=\int_{0}^{\infty}\delta(\tau)P(r,\tau)d\tau+K_1 M\left(Q^C(K_2),\tfrac{1}{2};\psi(C)\right)M\left(Q^r(K_2),S^r;\varsigma r\right)e^{\varphi r} \tag{17}$$

Where, P(r,T) is the price of a default-free zero-coupon bond with time-to-maturity of T, M(a,b;u) is the Kummer function[6], and K_i are constants.

By CIR, for P(r,t) we have:

$$P(r,T)=A(T)\exp\{-B(T)\,r\} \tag{18}$$

where;

$$\begin{aligned}
A(T) &\equiv \left[\frac{2\gamma\, e^{\frac{1}{2}(\kappa+\lambda*_r+\gamma)t}}{(\kappa+\lambda*_r+\gamma)(e^{\gamma T}-1)+2\gamma}\right]^{\frac{2\kappa\theta}{\sigma_r^2}} \\
B(T) &\equiv \frac{2\gamma\, e^{\frac{1}{2}\gamma T}}{(\kappa+\lambda*_r+\gamma)(e^{\gamma T}-1)+2\gamma} \\
\gamma &\equiv \left[(\kappa+\lambda*_r)^2+2\sigma_r^2\right]^{\frac{1}{2}}
\end{aligned} \tag{19}$$

where, λ^*_r is risk premium parameter of a default-free zero-coupon bond with respect to instantaneous rate, r. Furthermore, the parameters are given by:

$$\begin{aligned}
Q^C(K_2) &\equiv -\frac{K_2}{2\alpha} \\
\psi(C) &\equiv -\frac{\alpha}{\sigma^2}\left(C-\frac{\lambda_C\sigma^2}{\alpha}\right)^2 \\
S^r &\equiv \frac{2\kappa\theta}{\sigma_r^2\varsigma} \\
Q^r(K_2) &\equiv -\frac{\varphi}{\varsigma}S-\frac{2K_2}{\sigma_r^2\varsigma^2} \\
\varphi &\equiv \frac{\kappa+\lambda_r\sigma_r^2}{\sigma_r^2}-\frac{1}{2}\varsigma
\end{aligned} \tag{20}$$

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[6] M(a,b;z) are defined respectively as following [see Abromowitz and Stegun (1977) for details]

$$M(a,b;z)=1+\frac{az}{b}+\frac{(a)_2 z^2}{(b)_2 2!}+\cdots+\frac{(a)_n z^n}{(b)_n n!}+\cdots$$

where;

$$(a)_n=a(a+1)(a+2)\cdots(a+n-1),\quad (a)_0=1$$

$$\varsigma \equiv 2\left[\left(\frac{\kappa+\lambda_r\sigma_r^2}{\sigma_r^2}\right)^2+\frac{2\left(1-\tau_p\right)}{\sigma_r^2}\right]^{\frac{1}{2}}$$

Proof. See Appendix A.

Applying the relevant boundary conditions, the net benefits of debt, $NB(C,r;\boldsymbol{X})$, can be expressed as following:

$$\begin{aligned} NB(C,r;\boldsymbol{X}) = & \left[\left(\tau_c-\tau_p\right)\iota-\left(1-\tau_p\right)\kappa_m mB-\kappa_F-\kappa_V B\right]\int_0^\infty P(r,\tau)d\tau \\ & +A_1 M\left(Q^C(A_2),\tfrac{1}{2};\psi(C)\right)M\left(Q^r(A_2),S^r;\varsigma r\right)e^{\varphi r} \end{aligned} \tag{21}$$

where corresponding to K_1 and K_2 , constants A_1 and A_2 can be determined using the boundary conditions of (9). A closed form solution for $\boldsymbol{A} \equiv (A_1 , A_2)$ is provided in Appendix B. Since the (21) now can be analytically expressed in terms of optimal capital structure, as well as parameters of cash flow and interest dynamics, the optimization problem can be rewritten as:

$$\begin{aligned} \max_{\boldsymbol{X}\,|\,\boldsymbol{\Phi}} \left\{NB(C,r;\boldsymbol{X})\right\} = \max_{\boldsymbol{X}\,|\,\boldsymbol{\Phi}} & \left\{\left[\left(\tau_c-\tau_p\right)\iota-\left(1-\tau_p\right)\kappa_m mB-\kappa_F-\kappa_V B\right]\int_0^\infty P(r,\tau)d\tau\right. \\ & \left.+A_1 M\left(Q^C(A_2),\tfrac{1}{2};\psi(C)\right)M\left(Q^r(A_2),S^r;\varsigma r\right)e^{\varphi r}\right\} \end{aligned} \tag{22}$$

S.T.

$$\begin{aligned} & D(C^U,r^U;\boldsymbol{X}) = (1+\beta)B \\ & \lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U,\boldsymbol{R}^U;\boldsymbol{X})/\partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U,\boldsymbol{R}^U;\boldsymbol{X})/\partial \boldsymbol{R} = 0 \\ & D(C^L,\boldsymbol{R}^L;\boldsymbol{X}) = (1-g)v(C^L,\boldsymbol{R}^L) \\ & \lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L,\boldsymbol{R}^L;\boldsymbol{X})/\partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L,\boldsymbol{R}^L;\boldsymbol{X})/\partial \boldsymbol{R} = 0 \end{aligned}$$

Though the solution to the constrained maximization problem cannot be obtained analytically in terms of cash flow and interest rates dynamics, using numerical optimization procedures, one can easily solve for the optimal capital structure, $\boldsymbol{X}$.

Since the unlevered value of debt is also a contingent claim on cash flows and interest rate one may have to solve similar partial differential equation to (6). However, from our familiar discounted cash flow valuation models, one can write:

$$v(C^L, r^L; \boldsymbol{X}) = \int_0^\infty \mathbf{E}_Q\left[C(\tau)\middle| C(0) = C^L\right] P(r^L, \tau)\, d\tau$$

where, $P(r^L,\tau)$ is the value of a zero-coupon bond defined by (18) and (19), and $C^L \equiv \iota + mB$, is the expected cash flow at default, and defined as following:

$$\mathbf{E}_Q\left[C(t)\middle|C(0) = C^L\right] = C^L e^{\alpha t} + \frac{\lambda_C \sigma}{\alpha}\left(1 - e^{\alpha t}\right)$$

A. Value of Corporate Debt and Credit Spread

Given the optimal capital structure found in previous section, we know study the comparative statics of the corporate debt with respect to

III. Optimal Capital-Maturity Structure and Term Structure of Interest Rates

Though in previous section, we have found some startling results, one could argue that without accounting for complexity of term structure of interest, one cannot examine the effect of the dynamics of interest rate accurately. Despite significant advances in the literature of term structure of interest rates, there are no correct models. However, recent studies of the term structure of interest rates have shown that the explanatory power of the model increases as the number of factors increases. [Brenan and Schwartz (1979), Schaefer and Schwartz (1984), Heath, Jarrow and Morton (1988), Dybvig (1989) and, Lamoureux and Witte (1998)]. Brenan and Schwartz (1979) suggested a two-factor model, using long-term as well as short-term interest rates. Schaefer and Schwartz (1984) used the spread between short-term and long-term interest rates and long-term interest rates to model the term structure of interest rates. Litterman and Scheinkman (1988) proposed that factors such as level, steepness, and curvature of the yield curve could be used to explain the variations of the yield curve. Lamoureux and Witte (1998) showed that as number of factors increases the explanatory power of the model increases.

Chen (1996) presented a three-factor affine model of the term structure of interest rates, which revealed similar properties to that of Litterman and Scheinkman (1988). Dai and Singleton (1998) found that in a comparison among different affine models of the term structure of interest rates, Chen's (1996) performed the best. Hence, following Chen (1996), we model the evolution of the term structure of interest rates, $\boldsymbol{R} \equiv (r,\theta,\upsilon)^{\mathsf{T}}$, is governed by three factors for which the dynamics of instantaneous rates, r, central tendency, θ, and volatility of instantaneous rates, υ, is defined by the system of stochastic differential equation of:

$$\begin{aligned} dr(t) &= \kappa[\theta(t) - r(t)]dt + \sqrt{\upsilon(t)}\, dW^r \\ d\theta(t) &= \xi[\bar{\theta} - \theta(t)]dt + \zeta\sqrt{\theta(t)}\, dW^\theta \\ d\upsilon(t) &= \varpi[\bar{\upsilon} - \upsilon(t)]dt + \eta\sqrt{\upsilon(t)}\, dW^\upsilon \end{aligned} \tag{23}$$

and, $W^r \perp W^\theta \perp W^\upsilon$ are Weiner processes of the order one.

Assuming the term structure of interest rate can be modeled appropriately by (23), and cash flows of the firm can also be modeled by (15), under technical conditions, the value of a contingent claim, *F*(*C*,***R***;***X***), on firm's cash flows, *C*, and term structure of interest rates, ***R***, which perpetually pays a known amount, δ, can be expressed as:

$$\begin{aligned}\tfrac{1}{2}\sigma^2 F_{CC} + \left(\alpha C - \lambda_C \sigma^2\right)F_C + \tfrac{1}{2}\upsilon F_{rr} + \left[\kappa\theta - \kappa r - \lambda_r \upsilon\right]F_r + \tfrac{1}{2}\zeta^2 \theta F_{\theta\theta} + \left[\xi\bar{\theta} - \left(\xi + \lambda_\theta \zeta^2\right)\theta\right]F_\theta \\ + \tfrac{1}{2}\eta^2 \upsilon F_{\upsilon\upsilon} + \left[\varpi\bar{\upsilon} - \left(\varpi + \lambda_\upsilon \eta^2\right)\upsilon\right]F_\upsilon - r\left(1-\tau_p\right)F + \delta(t) = 0\end{aligned} \tag{26}$$

Using change of variables such as $C \equiv C, \quad u \equiv (\theta - r)\upsilon^{\frac{1}{2}}, \quad \theta \equiv \theta, \quad and \quad \upsilon \equiv \upsilon$, equation (26) can be rewritten as:

$$\begin{aligned}\tfrac{1}{2}\sigma^2 F_{CC} + \left(\alpha C - \lambda_C \sigma^2\right)F_C + \tfrac{1}{2} F_{uu} - \left[\kappa u - \lambda_r \sqrt{\upsilon}\right]F_u + \tfrac{1}{2}\zeta^2 \theta F_{\theta\theta} + \left[\xi\bar{\theta} - \left(\xi + \lambda_\theta \zeta^2\right)\theta\right]F_\theta \\ - \left(1-\tau_p\right)\theta F + \tfrac{1}{2}\eta^2 \upsilon F_{\upsilon\upsilon} + \left[\varpi\bar{\upsilon} - \left(\varpi + \lambda_\upsilon \eta^2\right)\upsilon\right]F_\upsilon + \left(1-\tau_p\right)\sqrt{\upsilon}u F + \delta(t) = 0\end{aligned}$$

Applying the change of variables of $C \equiv C, \quad w \equiv \kappa u - \lambda_r \upsilon, \quad \theta \equiv \theta, \quad and \quad \upsilon \equiv \upsilon$, then we can rewrite the equation as following:

$$\begin{aligned}\tfrac{1}{2}\sigma^2 F_{CC} + \left(\alpha C - \lambda_C \sigma^2\right)F_C + \tfrac{1}{2}\kappa^2 F_{ww} - \kappa w F_w + \tfrac{1}{\kappa}\left(1-\tau_p\right)w\upsilon^{\frac{1}{2}} F \\ + \tfrac{1}{2}\zeta^2 \theta F_{\theta\theta} + \left[\xi\bar{\theta} - \left(\xi + \lambda_\theta \zeta^2\right)\theta\right]F_\theta - \left(1-\tau_p\right)\theta F \\ + \tfrac{1}{2}\eta^2 \upsilon F_{\upsilon\upsilon} + \left[\varpi\bar{\upsilon} - \left(\varpi + \lambda_\upsilon \eta^2\right)\upsilon\right]F_\upsilon - \tfrac{1}{\kappa}\left(1-\tau_p\right)\lambda_r \upsilon F + \delta = 0\end{aligned}$$

Rearranging the equation, we can write (26) in form of operators ***L*** and ***V*** as following:

$$\boldsymbol{L}\left[C, w, \theta, \upsilon; \partial C, \partial w, \partial\theta, \partial\upsilon\right]F(C, w, \theta, \upsilon) = \boldsymbol{V}\left[C, w, \theta, \upsilon\right]F(C, w, \theta, \upsilon) - \delta \tag{27}$$

Subject to boundary conditions:

$$\begin{aligned}F(C^U, w^U, \theta^U, \upsilon^U) + K^U F(C_0, w_0, \theta_0, \upsilon_0) = B^U \\ F(C^L, w^L, \theta^L, \upsilon^L) + K^L F(C_0, w_0, \theta_0, \upsilon_0) = B^L\end{aligned}$$

where;

$$\begin{aligned}\boldsymbol{L}[\,\cdot\,] \equiv \tfrac{1}{2}\sigma^2 \partial_{CC} + \left(\alpha C - \lambda_C \sigma^2\right)\partial_C + \tfrac{1}{2}\kappa^2 \partial_{ww} - \kappa w \partial_w + \tfrac{1}{\kappa}\left(1-\tau_p\right)w \\ + \tfrac{1}{2}\zeta^2 \theta \partial_{\theta\theta} + \left[\xi\bar{\theta} - \left(\xi + \lambda_\theta \zeta^2\right)\theta\right]\partial_\theta - \left(1-\tau_p\right)\theta \\ + \tfrac{1}{2}\eta^2 \upsilon \partial_{\upsilon\upsilon} + \left[\varpi\bar{\upsilon} - \left(\varpi + \lambda_\upsilon \eta^2\right)\upsilon\right]\partial_\upsilon - \tfrac{1}{\kappa}\left(1-\tau_p\right)\lambda_r \upsilon \\ \boldsymbol{V}[\,\cdot\,] \equiv \tfrac{1}{\kappa}\left(1-\tau_p\right)w\left(1-\sqrt{\upsilon}\right)\end{aligned} \tag{28}$$

In order to find a solution to (27), we can use the Green's function that solves following:

$$\boldsymbol{L}[G(C, w, \theta, \upsilon; \hat{C}, \hat{r}, \hat{\theta}, \hat{\upsilon})] = 0 \tag{29}$$

Subject to boundary conditions:

$$G(C^U, w^U, \theta^U, \upsilon^U; \hat{C}, \hat{r}, \hat{\theta}, \hat{\upsilon}) + K^U\, G(C_0, w_0, \theta_0, \upsilon_0; \hat{C}, \hat{r}, \hat{\theta}, \hat{\upsilon}) = B^U \delta\left(C - \hat{C}\right)\delta(w - \hat{w})\delta\left(\theta - \hat{\theta}\right)\delta(\upsilon - \hat{\upsilon})$$
$$G(C^L, w^L, \theta^L, \upsilon^L; \hat{C}, \hat{r}, \hat{\theta}, \hat{\upsilon}) + K^L\, G(C_0, w_0, \theta_0, \upsilon_0; \hat{C}, \hat{r}, \hat{\theta}, \hat{\upsilon}) = B^L \delta\left(C - \hat{C}\right)\delta(w - \hat{w})\delta\left(\theta - \hat{\theta}\right)\delta(\upsilon - \hat{\upsilon})$$

where $\delta(\cdot)$ is the Dirac function.

With the method of separation of variables, we then solve for Green's functions and the value of contingent claim.

Lemma 1. *The Green's function for the fundamental PDE (26) is given by:*

$$\begin{aligned} G(C, w, \theta, \upsilon; \hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon}; \boldsymbol{X}) = K_1\,(\hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon})\, & M\left(Q^C\,(K_2\,(\hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon})), \tfrac{1}{2}; \psi(C)\right) \\ & \times M\left(Q^w\,(K_2\,(\hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon})), \tfrac{1}{2}; \varsigma(w)\right) \\ & \times M(Q^\theta, S^\theta; \varsigma^\theta \theta)\, M(Q^\upsilon, S^\upsilon; \varsigma^\upsilon \upsilon) \\ & \times \exp\left\{\left(1 - \tau_p\right)\left[(r - \theta)\upsilon^{-\frac{1}{2}} + \lambda_r \upsilon\right] + \varphi^\theta \theta + \varphi^\upsilon \upsilon\right\} \end{aligned} \tag{30}$$

where,

$$\begin{aligned} & Q^C\,(K_2\,(\hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon})) \equiv -\frac{K_2\,(\hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon})}{2\alpha} \\ & \psi(Q) \equiv -\frac{\alpha}{\sigma^2}\left(C - \frac{\lambda_C \sigma^2}{\alpha}\right)^2 \\ & \varsigma \equiv -\left(f + g\,r\right)^2 / 2g \\ & f \equiv -\frac{2}{\kappa} \\ & g \equiv \frac{1}{\kappa^2}\left[\left(\frac{1 - \tau_p}{\kappa}\right)^2 + 2\,K_2\,(\hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon})\right] \\ & Q^w \equiv -\frac{K_2\,(\hat{C}, \hat{w}, \hat{\theta}, \hat{\upsilon})}{2\,\kappa} - \frac{\left(1 - \tau_p\right)^2}{4\kappa^3} \\ & Q^\theta \equiv -\frac{\varphi^\theta}{\varsigma^\theta} S^\theta \\ & S^\theta \equiv \frac{2\xi\overline{\theta}}{\zeta^2 \varsigma^\theta} \\ & \varphi^\theta \equiv \frac{\xi + \lambda_\theta \zeta^2}{\zeta^2} - \frac{1}{2}\varsigma^\theta \end{aligned} \tag{31}$$

$$\varsigma^{\theta} \equiv 2\left[\left(\frac{\xi+\lambda_{\theta}\zeta^{2}}{\zeta^{2}}\right)^{2}+\frac{2(1-\tau_{p})}{\zeta^{2}}\right]^{\frac{1}{2}}$$

$$Q^{\upsilon} \equiv -\frac{\varphi^{\upsilon}}{\varsigma^{\upsilon}} S^{\upsilon}$$

$$S^{\upsilon} \equiv \frac{2\varpi\,\overline{\upsilon}}{\eta^{2}\varsigma^{\upsilon}}$$

$$\varphi^{\upsilon} \equiv \frac{\varpi+\lambda_{\upsilon}\eta^{2}}{\eta^{2}}-\frac{1}{2}\varsigma^{\upsilon}$$

$$\varsigma^{\upsilon} \equiv 2\left[\left(\frac{\varpi+\lambda_{\upsilon}\eta^{2}}{\eta^{2}}\right)^{2}+\frac{2(1-\tau_{p})\lambda_{r}}{\kappa\eta^{2}}\right]^{\frac{1}{2}}$$

where, $M(a,b;u)$ is the Kummer function, and K_i are constant that solve following boundary conditions:

$$G(C^{U},w^{U},\theta^{U},\upsilon^{U};\hat{C},\hat{r},\hat{\theta},\hat{\upsilon})+K^{U}\,G(C_{0},w_{0},\theta_{0},\upsilon_{0};\hat{C},\hat{r},\hat{\theta},\hat{\upsilon})=B^{U}\delta\left(C-\hat{C}\right)\delta(w-\hat{w})\delta\left(\theta-\hat{\theta}\right)\delta(\upsilon-\hat{\upsilon})$$
$$G(C^{L},w^{L},\theta^{L},\upsilon^{L};\hat{C},\hat{r},\hat{\theta},\hat{\upsilon})+K^{L}\,G(C_{0},w_{0},\theta_{0},\upsilon_{0};\hat{C},\hat{r},\hat{\theta},\hat{\upsilon})=B^{L}\delta\left(C-\hat{C}\right)\delta(w-\hat{w})\delta\left(\theta-\hat{\theta}\right)\delta(\upsilon-\hat{\upsilon})$$

where, $K^U \equiv -\rho^U$, $K^L \equiv -\rho^L$, $B^U \equiv -\rho^U\,\kappa_V B - \kappa_F$, and, $B^L \equiv -\rho^L\,\kappa_V B - \kappa_F - gB - G$.

Using the perturbation method of approximation, we can derive a closed form solution for the value of the contingent claim.

Proposition 2. *Assume an economy in which the term structure of interest rates follows a dynamics such as (22) and a firm's cash flow can be defined properly by an Ornstein-Uhlenbeck process such as (15). Given that the aforementioned processes are orthogonal, then the value of a contingent claim, $F(C,\boldsymbol{R};\boldsymbol{X})$, on firm's cash flows, C, and term structure of interest rates which perpetually pays a known amount, δ, can be expressed is:*

$$F(C,w,\theta,\upsilon)=F_{0}(C,w,\theta,\upsilon)+F_{1}(C,w,\theta,\upsilon)+F_{2}(C,w,\theta,\upsilon)+\cdots \tag{32}$$

where;

$$F_{0}(C,w,\theta,\upsilon)=\delta\int\int\int\int G(C,w,\theta,\upsilon;\hat{C},\hat{w},\hat{\theta},\hat{\upsilon})d\hat{C}\,d\hat{w}\,d\hat{\theta}\,d\hat{\upsilon}$$

and,

$$F_{1}(C,w,\theta,\upsilon)=\int\int\int\int G(C,w,\theta,\upsilon;\hat{C},\hat{w},\hat{\theta},\hat{\upsilon})\boldsymbol{V}\left[\hat{C},\hat{w},\hat{\theta},\hat{\upsilon}\right]F_{0}(\hat{C},\hat{w},\hat{\theta},\hat{\upsilon})\,d\hat{C}\,d\hat{w}\,d\hat{\theta}\,d\hat{\upsilon}$$

$$F_2(C,w,\theta,\upsilon) = \int\int\int\int d\hat{C}\, d\hat{w}\, d\hat{\theta}\, d\hat{\upsilon} \int\int\int\int d\hat{\hat{C}} d\hat{\hat{w}} d\hat{\hat{\theta}} d\hat{\hat{\upsilon}}\, G(C,w,\theta,\upsilon;\hat{C},\hat{w},\hat{\theta},\hat{\upsilon})$$

$$\times \boldsymbol{V}\left[\hat{C},\hat{w},\hat{\theta},\hat{\upsilon}\right] G(\hat{C},\hat{w},\hat{\theta},\hat{\upsilon};\hat{\hat{C}},\hat{\hat{w}},\hat{\hat{\theta}},\hat{\hat{\upsilon}}) \boldsymbol{V}\left[\hat{\hat{C}},\hat{\hat{w}},\hat{\hat{\theta}},\hat{\hat{\upsilon}}\right] F_0(\hat{\hat{C}},\hat{\hat{w}},\hat{\hat{\theta}},\hat{\hat{\upsilon}})$$

$$\vdots$$

where;

$$\boldsymbol{V}[C,w,\theta,\upsilon] = \frac{1-\tau_p}{\kappa} w\left(1-\upsilon^{\frac{1}{2}}\right)$$

Proof. See Appendix A.

Applying the relevant boundary conditions, and substituting δ cash flow the optimization problem can be rewritten as:

$$\max_{\boldsymbol{X}\,|\boldsymbol{\Phi}} \{NB(C,\boldsymbol{R};\boldsymbol{X})\}$$

$$\text{S.T.} \tag{22}$$

$$D(C^U, r^U; \boldsymbol{X}) = (1+\beta)B$$

$$\lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U,\boldsymbol{R}^U;\boldsymbol{X})/\partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U,\boldsymbol{R}^U;\boldsymbol{X})/\partial \boldsymbol{R} = 0$$

$$D(C^L,\boldsymbol{R}^L;\boldsymbol{X}) = (1-g)v(C^L,\boldsymbol{R}^L)$$

$$\lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L,\boldsymbol{R}^L;\boldsymbol{X})/\partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L,\boldsymbol{R}^L;\boldsymbol{X})/\partial \boldsymbol{R} = 0$$

Though the solution to maximization problem cannot be obtained analytically in terms of cash flow and interest rates dynamics, using numerical optimization procedures, one can easily solve for the optimal capital structure, ***X***.

IV. Optimal Priority Structure and Term Structure of Interest Rates

Though in previous section, we have focused primarily on the effect of factors of term structure of interest rates on capital as well as maturity structure. However, one the most important features of corporate debt structure is the priority hierarchy of the debt claims. Black and Cox (1976) have put forth the first analysis of such kind, and shown that in presence of junior claims, the senior debt's loss of value at bankruptcy decreases. Smith and Warner (1979) speak in length regarding the senior bond covenants. In accordance with their analysis and in addition to assumptions A0 – A3, we refine original model by following assumptions:

(A4′) The firm's capital consists of senior callable and junior callable debt, as well as equity. Both senior debt and junior claims are coupon bonds, with continuous coupon payments of ι^S and ι^J, par values of B^S and B^J, and average maturities of M^S, and M^J, respectively. The firm can also adjust its capital structure at any time, maintaining the priority rules. For both kind of debt claims, the recapitalization is costly: there exist fixed recapitalization costs, κ_F, as well as proportional costs, κ_V, on absolute adjustments of debt's face value adjustments, $|\Delta B|$. Hence, the total recapitalization cost is of the form $\kappa_F + \kappa_V\, |\Delta B|$.

(A5′) Senior debt covenants requires firm to retire both debt claims' face values continuously at constant rates of m^S and m^J, providing that $m^J \leq m^S$. In essence, this is equivalent of having two sinking funds for either debt classes, while the senior debt's sinking fund is funded at a faster pace. This assumption, indeed allows senior bondholders to assure full payment before junior claimants are paid in full. Like original model, we assume that the continuous retirement and reissuance of debt is costly. We assume that regardless of priority the cost is proportional at a rate κ_m to retirement rate, mB.

(A6′) The senior bond indenture provisions prohibits stockholders from selling assets to pay any cash distribution to junior claimants or stockholders, and maintains absolute priority for senior bondholders. In the event that the firm's cash flows are not sufficient enough to cover the required debt services, senior or junior, firm defaults. In the event of default, the senior bondholders have absolute priority, meaning that junior claimants cannot be paid unless senior creditors' principle is paid in full. If the market value of the unlevered assets in place is less than face value of senior debt, senior bondholders liquidate the assets for its market value and receive the payments. If, however, the market value of assets in place is greater than face value of senior debt but less than the face value of total debt, then after liquidation and full payment of par to senior claimants, the junior bondholders receive the residual value. For simplicity, we assume that the market value of assets in place is proportional to unlevered value of the firm with factor of K^*. Regardless of the outcome, the financial distress (default) is costly and the total deadweight cost is equal to $g\, v(C^L,\boldsymbol{R}^L)$, where $0 \leq g < 1$, and $v(C^L,\boldsymbol{R}^L)$ is the unlevered value of the firm at the time of default.

(A7′) If the value of corporate bond, senior or junior, is equal to some pre-specified call price, $B^{iC} = (1+\beta)B^i$, where β is known and fixed. This is indeed departs from reality since most call prices decrease in time. In such event, firm can recall the outstanding bonds and issue a new bond with new face value. For simplicity, we assume that at recapitalization, the optimal capital structure, $\boldsymbol{X} = (B^S, \iota^S, C^{SU}, \boldsymbol{R}^{SU}, \boldsymbol{R}^{SL}, m^S, B^J, \iota^J, C^{JU}, \boldsymbol{R}^{JU}, \boldsymbol{R}^{JL}, m^J)$, is adjusted with a factor of proportionality $\rho^U \equiv v(C^U,\boldsymbol{R}^U)/v(C_0,\boldsymbol{R}_0)$, where v is the unlevered value of the firm [see Goldstein et al (1997) and Leland (1998).]

(A8′) Additionally, we assume that senior debt indenture requires that face value and average maturity of the junior to be weakly smaller than those of the senior debt or, $B^J \leq B^S$ and $m^J \leq m^S$. Since junior bondholders provide funding at a disadvantage, we also assume that risk premia of the junior debt is greater than that of senior debt, or $\boldsymbol{\Lambda}^J > \boldsymbol{\Lambda}^S$.

A. Valuation of Total Net Debt Benefits and Boundary Conditions

Like original model, the value of any contingent claim on cash flows and term structure is given by (26). In presence of two types of debt, however, the total net debt benefit is rather complicated. As such, we define the total net debt benefits as a sum of net benefits of senior and junior debt as following:

$$NB(C, \boldsymbol{R}; \boldsymbol{X}) \equiv NB^S(C, \boldsymbol{R}; \boldsymbol{X}) + NB^J(C, \boldsymbol{R}; \boldsymbol{X}) \tag{25}$$

Since events proceedings a default is a function of senior and junior debt characteristics, the boundary conditions for each of the two classes of bonds are different. For upper boundary, call provision, the boundary conditions for both senior and junior debt is similar to that of the original model, correcting for each class's characteristics. Hence, for senior debt claims, $NB^S(C^{SU},\boldsymbol{R}^{SU};\boldsymbol{X}) = \rho^{SU} NB(C_0,\boldsymbol{R}_0;\boldsymbol{X}) + (\rho^{SU}+1)(\kappa_F + \kappa_V B^S)$, where $\rho^{SU} \equiv v(C^{SU},\boldsymbol{R}^{SU})/v(C_0,\boldsymbol{R}_0)$, and $D^S(C^{SU},\boldsymbol{R}^{SU};\boldsymbol{X}) \equiv B^{SC}$. Also, for junior debt claims, $NB^J(C^{JU},\boldsymbol{R}^{JU};\boldsymbol{X}) = \rho^{JU} NB(C_0,\boldsymbol{R}_0;\boldsymbol{X}) + (\rho^{JU}+1)(\kappa_F + \kappa_V B^J)$, where $\rho^{JU} \equiv v(C^{JU},\boldsymbol{R}^{JU})/v(C_0,\boldsymbol{R}_0)$, and $D^J(C^{JU},\boldsymbol{R}^{JU};\boldsymbol{X}) \equiv B^{JC}$.

For lower boundaries, we have to note that the tax benefits and recapitalization costs will be different across the senior and junior debt. Having considered the impact of such differences, for senior debt, we have that $NB^S(C^{SL},\boldsymbol{R}^{SL};\boldsymbol{X}) = -g\, v(C^{SL},\boldsymbol{R}^{SL})$. For junior debt claims, however, we have $NB^J(C^{JL},\boldsymbol{R}^{JL};\boldsymbol{X}) = -g\, v(C^{JL},\boldsymbol{R}^{JL}) - B^S$. Additionally we require that $D^S(C^{SL},\boldsymbol{R}^{SL};\boldsymbol{X}) \equiv (1-g)\, v(C^{SL},\boldsymbol{R}^{SL})$, $D^S(C^{JL},\boldsymbol{R}^{JL};\boldsymbol{X}) \equiv B^S$, $D^J(C^{SL},\boldsymbol{R}^{SL};\boldsymbol{X}) \equiv 0$, and $D^J(C^{JL},\boldsymbol{R}^{JL};\boldsymbol{X}) \equiv (1-g)\, v(C^{JL},\boldsymbol{R}^{JL})$.

B. Optimal Total Capital Structure and Term Structure of Interest Rates:

Since the objective of the firm is to maximize the total levered value, or equivalently to maximize the total net benefits of debt (i.e. tax benefits minus all costs), providing that (A8) is held. Thus, we can write the objective function as

$$\max_{\boldsymbol{X}|\Phi} \left\{ NB(C, \boldsymbol{R}; \boldsymbol{X}) \equiv NB^S(C, \boldsymbol{R}; \boldsymbol{X}) + NB^J(C, \boldsymbol{R}; \boldsymbol{X}) \right\} \tag{26}$$

subject to following constrains:

$$\begin{aligned}
&\boldsymbol{\Lambda}^J > \boldsymbol{\Lambda}^S \\
&B^S > B^J \\
&D(C^U, r^U; \boldsymbol{X}) = (1+\beta)B \\
&\lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U, \boldsymbol{R}^U; \boldsymbol{X})/\partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^U,\boldsymbol{R}^U\}} \partial D(C^U, \boldsymbol{R}^U; \boldsymbol{X})/\partial \boldsymbol{R} = 0 \\
&D(C^L, \boldsymbol{R}^L; \boldsymbol{X}) = (1-g)v(C^L, \boldsymbol{R}^L) \\
&\lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L, \boldsymbol{R}^L; \boldsymbol{X})/\partial C = \lim_{\{C,\boldsymbol{R}\}\to\{C^L,\boldsymbol{R}^L\}} \partial E(C^L, \boldsymbol{R}^L; \boldsymbol{X})/\partial \boldsymbol{R} = 0
\end{aligned}$$

where, $NB(C,\boldsymbol{R};\boldsymbol{X}) \equiv NB^S(C,\boldsymbol{R};\boldsymbol{X}) + NB^J(C,\boldsymbol{R};\boldsymbol{X})$, $\boldsymbol{\Phi} = (\phi^C, \boldsymbol{\phi}^{\boldsymbol{R}})$ is the market-clearing vector of the factors' risk premia for corporate debt, and $C^{JL} \le C^{SL} \le C \le C^U$ and $\boldsymbol{R}^{JL} \le \boldsymbol{R}^{SL} \le \boldsymbol{R} \le \boldsymbol{R}^U$.

V. Optimal Capital Structure and Inflation

To analyze the impact of inflation and parametric structure of the its dynamics on the optimal capital-maturity choice and maintain tractability, we assume that term structure of interest rates can be appropriately modeled by a one-factor CIR model, or equivalently (14). The inflation, $\pi(t)$, and the firm's cash flows, $C(t)$ are both stochastic and they can be appropriately modeled as:

$$\begin{gathered} d\pi(t) = \kappa_\pi \left(\overline{\pi} - \pi(t)\right)\pi(t)\,dt + \sigma_\pi \pi^{\frac{3}{2}}(t) dW^{\pi} \\ dC(t) = \alpha\,\pi(t) C(t)\,dt + \sigma\,\pi^{\frac{1}{2}}(t)\,dW^{C} \end{gathered} \tag{28}$$

Like previous cases, we maintain that evolutions of inflation, cash flows and interest rates to independent. Hence, we know that correlation between the inflation and cash flow is zero.

Proposition 3. *Assuming the aforementioned economy, and that all processes are orthogonal, then the value of the value of a contingent claim, F(C,r,π;**X**), on firm's cash flows, C, interest rates, r, and inflation, π, which perpetually pays a known amount, δ, can be expressed as:*

$$\begin{aligned} &\tfrac{1}{2}\sigma^2 \pi\, F_{CC} + \left(\alpha\, C - \lambda_C \sigma^2\right)\pi\, F_C + \tfrac{1}{2}\sigma_r^2\, r F_{rr} + \left[\kappa\theta - \left(\kappa + \lambda_r\right) r\right] F_r \\ &\quad + \tfrac{1}{2}\sigma_\pi^2 \pi^3 F_{\pi\pi} + \left[\left(\kappa_\pi \overline{\pi} - \lambda_\pi^1 \sigma_\pi^2\right) - \left(\kappa_\pi + \lambda_\pi^1\right)\pi\right]\pi F_r - r\left(1 - \tau_p\right) F + \delta(t) = 0 \end{aligned} \tag{29}$$

*where, λ_r is risk premium parameter of the contingent claim F(C,r,π;**X**) with respect to interest rates, λ_C is risk premium parameter of the contingent claim F(C,r,π;**X**) with respect to cash flows. For which there exist a general solution, F(C,r,π;**X**) of the form:*

$$F\left(C, r, \pi; \boldsymbol{X}\right) = \int_0^\infty \delta(\tau) P\left(r, \pi, \tau\right) d\tau + K_1\, M\left(Q^C(K_2), \tfrac{1}{2}; \psi(C)\right) M\left(Q^r, S^r; \varsigma\, r\right) M\left(Q^v, S^v; v\right) v^\eta\, e^{\varphi r} \tag{30}$$

where, P(r,π,t) is the price of a default-free zero-coupon bond with time-to-maturity of T, M(Q,S;u) is the Kummer function, and K_i are constants.

By CIR (1985) and Ahn and Gao (1999), for P(r,π,t) we have:

$$P(r, \pi, T) = A(T)\,\Pi^{Q}\, M\left(Q, S; \Pi\right) \exp\left\{- B(T)\, r\right\} \tag{31}$$

where;

$$\begin{aligned} A(T) &\equiv \frac{\Gamma(S-Q)}{\Gamma(S)}\left[\frac{2\gamma\, e^{\frac{1}{2}\left(\kappa + \lambda*_r + \gamma\right)t}}{\left(\kappa + \lambda*_r + \gamma\right)\left(e^{\gamma T} - 1\right) + 2\gamma}\right]^{\frac{2\kappa\theta}{\sigma_r^2}} \\ B(T) &\equiv \frac{2\gamma\, e^{\frac{1}{2}\gamma T}}{\left(\kappa + \lambda*_r + \gamma\right)\left(e^{\gamma T} - 1\right) + 2\gamma} \\ \gamma &\equiv \left[\left(\kappa + \lambda*_r\right)^2 + 2\sigma_r^2\right]^{\frac{1}{2}} \\ \Pi &\equiv \frac{a(T)}{\pi} \\ a(T) &\equiv \frac{\kappa_\pi \overline{\pi} - \lambda_{\pi,1}}{e^{\left(\kappa_\pi \overline{\pi} - \lambda_{\pi,1}\right)T} - 1} \end{aligned} \tag{32}$$

$$
\begin{aligned}
&Q \equiv \left(\phi^{2}+2\right)^{\frac{1}{2}}-\phi \\
&S \equiv \frac{2\left(\kappa_{\pi}+\lambda *_{\pi,2}+\left(1+Q\right)\sigma_{\pi}^{2}\right)}{\sigma_{\pi}^{2}}
\end{aligned}
$$

and;

$$
\phi \equiv \frac{\kappa_{\pi}+\lambda *_{\pi,2}}{\sigma_{\pi}^{2}}+\frac{1}{2}
$$

where, λ^{*}_{r} is risk premium parameter of a default-free zero-coupon bond with respect to instantaneous rate, r, and $\lambda^{*}_{\pi,1}$ and $\lambda^{*}_{\pi,2}$ are the risk premium parameters of a default-free zero-coupon bond with respect to inflation, π.

The functional parameters of (30) are given by:

$$
\begin{aligned}
&Q^{C}(K_{2}) \equiv -\frac{K_{2}}{2\alpha} \\
&\psi(C) \equiv -\frac{\alpha}{\sigma^{2}}\left(C-\frac{\lambda_{C}\sigma^{2}}{\alpha}\right)^{2} \\
&S^{r} \equiv \frac{2\kappa\theta}{\sigma_{r}^{2}\varsigma} \\
&Q^{r}(K_{2}) \equiv -\frac{\varphi}{\varsigma}S \\
&\varphi \equiv \frac{\kappa+\lambda_{r}\sigma_{r}^{2}}{\sigma_{r}^{2}}-\frac{1}{2}\varsigma \\
&\varsigma \equiv 2\left[\left(\frac{\kappa+\lambda_{r}\sigma_{r}^{2}}{\sigma_{r}^{2}}\right)^{2}+\frac{2\left(1-\tau_{p}\right)}{\sigma_{r}^{2}}\right]^{\frac{1}{2}} \\
&\nu \equiv \frac{\Delta}{\pi} \\
&\Delta \equiv \frac{\sigma_{\pi}^{2}}{2\pi *} \\
&Q^{\nu} \equiv \eta \\
&S^{\nu} \equiv \frac{2\kappa *_{\pi}}{\sigma_{\pi}^{2}}+2\left(\eta+1\right) \\
&\eta \equiv -\left(\frac{\kappa *_{\pi}}{\sigma_{\pi}^{2}}+\frac{1}{2}\right)-\left[\left(\frac{\kappa *_{\pi}}{\sigma_{\pi}^{2}}+\frac{1}{2}\right)-\frac{2K_{1}}{\sigma_{\pi}^{2}}\right]^{\frac{1}{2}} \\
&\pi * = \kappa_{\pi}\bar{\pi}-\lambda_{\pi,1}\sigma_{\pi}^{2}
\end{aligned}
\tag{33}
$$

and;

$$\kappa *_{\pi} = \kappa_{\pi} + \lambda_{\pi,2}$$

where, λ_r is risk premium parameter of the contingent claim with respect to the instantaneous rate, r, and $\lambda_{\pi,1}$ and $\lambda_{\pi,2}$ are the risk premium parameters of the contingent claim with respect to the inflation, π.

Proof. See Appendix A.

VI. On the Effect of Jumps in Cash Flows and Interest Rates

Despite the appeal of diffusion models, one shortcoming of such approach in modeling dynamics of processes is that one cannot properly motivate significant likelihood of the sudden changes. Jump-diffusion models, however, have provided interesting alternative as to how dynamics of processes could potentially behave through time. To analyze the impact of sudden changes (discontinuity) of cash flows and risk-free rate on the optimal capital structure, we define the dynamics of cash flows and risk-free rate as jump-diffusion processes.

Following CIR, we define the term structure as following

$$dr(t) = \kappa\left(\theta - r(t)\right)dt + \sigma_r r^{\frac{1}{2}}(t)\, dW^r + J^r\left(\mu_r, \gamma_r^2\right)d\vartheta^r(h_r) \tag{35}$$

where, $r(t)$ is the risk-free rate, θ is the central tendency of the risk-free rate, σ is the instantaneous volatility parameter of short rates, J is the jump in short rates characterized by μ_r and γ_r^2, and ϑ^r is the Poisson arrival probability with parameter h_r.

Lemma 2. *Under the aforementioned term structure, the price of a default free bond is defined by following fundamental PDE:*

$$\tfrac{1}{2}\sigma_r^2\, r\, P_{rr} + \left[\kappa\theta - \left(\kappa + \lambda_r\right)r\right]P_r + h\,\boldsymbol{E}\left[P(r+J^r) - P(r)\right] - rP + P_T \equiv 0 \tag{36}$$

For which there exist a solution of the form:

$$P(r,T) = A(T)\exp\left\{-B(T)\,r\right\} \tag{37}$$

where;

$$\begin{aligned} A(T) &\equiv \exp\left\{\frac{4\Delta\left(e^{\gamma T}-1\right)}{\sigma_r^2\left[\left(\kappa+\lambda *_r+\gamma\right)\left(e^{\gamma T}-1\right)+2\gamma\right]} + \frac{2\left(\gamma-\kappa-\lambda *_r\right)\Omega - 4\gamma^2\Delta}{\left(\gamma-\kappa-\lambda *_r\right)^2}T\right\}\times \\ &\quad \left[\frac{1}{2}\left(\frac{\kappa+\lambda *_r}{\gamma}+1\right)\left(e^{\gamma T}-1\right)+1\right]^{\frac{4(\kappa+\lambda *_r)\Delta - 2\sigma_r^4\Omega}{\sigma_r^4}} \\ B(T) &\equiv \frac{2\gamma\, e^{\frac{1}{2}\gamma T}}{\left(\kappa+\lambda *_r+\gamma\right)\left(e^{\gamma T}-1\right)+2\gamma} \end{aligned} \tag{38}$$

$$
\begin{aligned}
&\gamma \equiv \left[(\kappa + \lambda *_r)^2 + 2\sigma_r^2 \right]^{\frac{1}{2}} \\
&\Delta \equiv \tfrac{1}{2} h_r \left(\mu_r^2 + \gamma_r^2 \right)
\end{aligned}
$$

and,

$$\Omega \equiv -\left(\kappa\, \theta + h_r\, \mu_r \right)$$

where, $\lambda^*{}_r$ is risk premium parameter of a default-free zero-coupon bond with respect to instantaneous rate, r.

Proof. See Appendix A.

Additionally, we define the cash flow process to evolve through time by:

$$dC(t) = \alpha\, C(t)\, dt + \sigma\ dW^C + J^C\left(\mu_C, \gamma_C^2 \right) d\vartheta^C (h_C) \tag{39}$$

where, $C(t)$ is the risk-free rate, α is the constant growth of cash flows, σ^2 is the constant variances of cash flows, J is the jump in cash flows characterized by μ_C and $\gamma_C{}^2$, and ϑ^C is the Poisson arrival probability with parameter h_C.

By Ito's lemma, the stochastic differential equation that determines the value of a contingent claim $F(C,r;\boldsymbol{X})$ which pays perpetually a known payoff of $\delta(t)$, is given by

$$
\begin{aligned}
dF = &\left[\tfrac{1}{2}\sigma^2\ F_{CC} + \alpha\, C\, F_C + \tfrac{1}{2}\sigma_r^2\ r F_{rr} + \kappa(\theta - r) F_r \right] dt + \sigma\ F_C dW^C + \sigma_r r^{\frac{1}{2}} F_r dW^r \\
&+ \left[F\left(C + J^C, r \right) - F(C, r) \ \middle|\ J^C \right] d\vartheta^C + \left[F\left(C, r + J^r \right) - F(C, r) \ \middle|\ J^r \right] d\vartheta^r
\end{aligned}
\tag{40}
$$

Assuming that risk premium on each primitive is proportional to the variance of the each primitive, the fundamental pricing equation can be expressed by a partial differential-difference equation, such as following:

$$
\begin{aligned}
&\tfrac{1}{2}\sigma^2\ F_{CC} + \left(\alpha\, C - \lambda_C \sigma^2 \right) F_C + \tfrac{1}{2}\sigma_r^2\ r\, F_{rr} + \left[\kappa\theta - \left(\kappa + \lambda_r \sigma_r^2 \right) r \right] F_r \\
&\quad + h_C\ \boldsymbol{E}\left[F\left(C + J^C, r \right) - F(C, r) \right] + h_r\ \boldsymbol{E}\left[F\left(C, r + J^r \right) - F(C, r) \right] - \left(1 - \tau_p \right) r F + \delta(t) \equiv 0
\end{aligned}
\tag{41}
$$

Using Taylor's approximations, the aforementioned partial differential-difference equation can be approximated by the following linear partial differential equation:

$$
\begin{aligned}
&\tfrac{1}{2} h_C \left(C^2 + 2\mu_C C + \mu_C^2 + \gamma_C^2 + \sigma^2 / h_C \right) F_{CC} + \left((\alpha + h_C) C - \lambda_C \sigma^2 + h_C \mu_C \right) F_C \\
&\quad + \tfrac{1}{2} h_r \left(r^2 + 2\mu_r r + \mu_r^2 + \gamma_r^2 + \sigma_r^2 r / h_r \right) F_{rr} + \left[\kappa\theta + h_r \mu_r - \left(\kappa + \lambda_r \sigma_r^2 - h_r \right) r \right] F_r \\
&\quad - \left(1 - \tau_p \right) r F + \delta(t) \equiv 0
\end{aligned}
\tag{42}
$$

Having changed variables as $\hat{C} \equiv C + \mu_C$ and $\tilde{r} \equiv r + \mu_r + \sigma_r^2 / 2h_r$, and rearranging the equation, we can write (43) in form of linear differential operators $\boldsymbol{L}$ and $\boldsymbol{V}$ as following:

$$\boldsymbol{L}\left[\hat{C}, \hat{r}; \partial\hat{C}, \partial\hat{r}\right] F(\hat{C}, \hat{r}) = \boldsymbol{V}\left[\hat{C}, \hat{r}\right] F(\hat{C}, \hat{r}) - \delta \tag{43}$$

Subject to boundary conditions:

$$\begin{aligned} &F(\hat{C}^{U}, \hat{r}^{U}) + K^{U}\ F(\hat{C}_{0}, \hat{r}_{0}) = B^{U} \\ &F(\hat{C}^{L}, \hat{r}^{L}) + K^{L}\ F(\hat{C}_{0}, \hat{r}_{0}) = B^{L} \end{aligned}$$

where;

$$\begin{aligned} &\boldsymbol{L}[\cdot] \equiv \left(\hat{C}^{2} + \Delta^{2}\right)\partial_{\hat{C}\hat{C}} + \left(\hat{\alpha}\,\hat{C} - \hat{\mu}\right)\partial_{\hat{C}} + \left(\hat{r}^{2} + \boldsymbol{\Omega}^{2}\right)\partial_{\hat{r}\hat{r}} + \left(\Theta - \Lambda\,\hat{r}\right)\partial_{\hat{r}} + \Psi^{*} \\ &\boldsymbol{V}[\cdot] \equiv \frac{2\,(1-\tau_{p})}{h_{r}}\,\hat{r} \end{aligned} \tag{44}$$

and,

$$\begin{aligned} &\Delta \equiv \left[\gamma_{C}^{2} + \frac{\sigma^{2}}{h_{C}}\right]^{\frac{1}{2}} \\ &\hat{\alpha} \equiv \frac{2\,\alpha}{h_{C}} + 2 \\ &\hat{\mu} \equiv \frac{2}{h_{C}}\left(\alpha\,\mu_{C} + \lambda_{C}\,\sigma^{2}\right) \\ &\Omega \equiv \gamma_{r}^{2} - \frac{\sigma_{r}^{2}}{h_{r}}\left(\mu_{r} + \frac{\sigma_{r}^{2}}{4\,h_{r}}\right) \\ &\Theta \equiv \left[\kappa\,\theta - \sigma_{r}^{2} + \left(\kappa + \lambda_{r}\,\sigma_{r}^{2}\right)\left(\mu_{r} + \frac{\sigma_{r}^{2}}{h_{r}}\right)\right]\frac{2}{h_{r}} \\ &\Lambda \equiv \left(\kappa + \lambda_{r}\,\sigma_{r}^{2} - h_{r}\right)\frac{2}{h_{r}} \\ &\Psi^{*} \equiv \frac{2}{h_{r}}\left(1 - \tau_{p}\right)\left(\mu_{r} + \frac{\sigma_{r}^{2}}{2\,h_{r}}\right) \end{aligned} \tag{45}$$

In order to find a solution to (43), we can use the Green's function that solves following:

$$\boldsymbol{L}[G(\hat{C}, \hat{r}; \hat{\hat{C}}, \hat{\hat{r}})] = 0$$

Subject to boundary conditions:

$$G(\hat{C}^{U}, \hat{r}^{U}; \hat{\hat{C}}, \hat{\hat{r}}) + K^{U}\ G(\hat{C}_{0}, \hat{r}_{0}; \hat{\hat{C}}, \hat{\hat{r}}) = B^{U}\,\delta\!\left(\hat{C} - \hat{\hat{C}}\right)\delta\!\left(\hat{r} - \hat{\hat{r}}\right)$$

$$G(\hat{C}^L, \hat{r}^L; \hat{\hat{C}}, \hat{\hat{r}}) + K^L\ G(\hat{C}_0, \hat{r}_0; \hat{\hat{C}}, \hat{\hat{r}}) = B^L \delta\left(\hat{C} - \hat{\hat{C}}\right) \delta\left(\hat{r} - \hat{\hat{r}}\right)$$

where $\delta(\cdot)$ is the Dirac function.

With the method of separation of variables, we then solve for Green's functions and the value of contingent claim.

Lemma 3. *The Green's function for the fundamental PDE (26) is given by:*

$$\begin{aligned} G\left(\hat{C}, \hat{r}; \hat{\hat{C}}, \hat{\hat{r}}\right) \equiv \left(\hat{C}^2 + \Delta^2\right)^{\frac{1}{2} - \frac{1}{4}\hat{\alpha}}\ P_\nu^\mu\left(i\, \hat{C}/\Delta\right) \exp\left(-\frac{\hat{\mu}\ \arctan(\hat{C}/\Delta)}{2\,\Delta}\right) \\ \times \left(\hat{r}^2 + \Omega^2\right)^{\frac{1}{2} + \frac{1}{4}\Lambda}\ P_{\bar{\nu}}^{\bar{\mu}}\left(i\, \hat{r}/\Omega\right) \exp\left(-\frac{\Theta\ \arctan(\hat{r}/\Omega)}{2\,\Omega}\right) \end{aligned} \tag{46}$$

where, P(r,T) is the price of a default-free zero-coupon bond with time-to-maturity of T, and $P_\nu^\mu(\cdot)$ *is the Legendre function, and parameters are defined as following:*

$$\begin{aligned} \nu &\equiv \tfrac{1}{2}\left[\left(\hat{\alpha} - 2\,\hat{\alpha} + 4\,K^* + 1\right)^{\frac{1}{2}} - 1\right] \\ \bar{\nu} &\equiv \tfrac{1}{2}\left[\left(-\Lambda + 2\,\Lambda - 4\,\Psi + 1\right)^{\frac{1}{2}} - 1\right] \\ \mu &\equiv \tfrac{1}{2}\left[\left(\hat{\alpha} - 2 - i\,\frac{\hat{\mu}}{2\,\Delta}\right)^2 - \frac{3\,\hat{\mu}^2}{2\,\hat{\Delta}^2}\right]^{\frac{1}{2}} \\ \bar{\mu} &\equiv \tfrac{1}{2}\left[\left(-\Lambda - 2 + i\,\frac{\Theta}{2\,\Delta}\right)^2 - \frac{3\,\Theta^2}{2\,\hat{\Delta}^2}\right]^{\frac{1}{2}} \\ K^* &\equiv \frac{2\,K_2}{h_C} \\ \Psi &\equiv \frac{2}{h_r}\left[\left(1 - \tau_p\right)\left(\mu_r + \frac{\sigma_r^2}{2\,h_r}\right) + K_2\right] \\ \Phi &\equiv \frac{2\left(1 - \tau_p\right)}{h_r} \end{aligned} \tag{47}$$

Proof. See Appendix A.

Like the case of the three-factor model, using the perturbation method of approximation, we can derive a closed form solution for the value of the contingent claim.

Proposition 4. *Assume an economy in which the term structure of interest rates follows a dynamics such as (35) and a firm's cash flow can be defined properly by a Jump Ornstein-Uhlenbeck process such as (39). Given that the aforementioned processes are orthogonal, then the value of a* *contingent claim, F(C,r;**X**), on firm's cash flows, C, and term structure of interest rates which perpetually pays a known amount, δ, can be expressed is:*

$$F(C,r) = F_0(C,r) + F_1(C,r) + F_2(C,r) + \cdots \tag{48}$$

where;

$$F_0(C,r) = \delta \int\int\int\int G(\hat{C},\hat{r};\hat{\hat{C}},\hat{\hat{r}})\, d\hat{\hat{C}}\, d\,\hat{\hat{r}}$$

and,

$$\begin{aligned} F_1(C,r) &= \int\int\int\int G(\hat{C},\hat{r};\hat{\hat{C}},\hat{\hat{r}})\boldsymbol{V}\left[\hat{\hat{C}},\hat{\hat{r}}\right] F_0(\hat{\hat{C}},\hat{\hat{r}})\, d\hat{\hat{C}}\, d\,\hat{\hat{r}} \\ F_2(C,w,\theta,\upsilon) &= \int\int\int\int d\hat{\hat{C}}\, d\,\hat{\hat{r}} \int\int\int\int d\tilde{C}\, d\,\tilde{r}\, G(\hat{C},\hat{r};\hat{\hat{C}},\hat{\hat{r}}) \\ &\qquad \times \boldsymbol{V}\left[\hat{\hat{C}},\hat{\hat{r}}\right] G(\hat{\hat{C}},\hat{\hat{r}};\tilde{C},\tilde{r})\boldsymbol{V}\left[\tilde{C},\tilde{r}\right] F_0(\tilde{C},\tilde{r}) \\ &\vdots \end{aligned}$$

where;

$$\boldsymbol{V}[C,r] = \frac{2(1-\tau_p)}{h_r} r$$

Proof. See Appendix A.

VII. Conclusion

Appendix A.

A.1. Proof of Proposition 1.

The solution to equation (16) consists of the present value of a continuous cash outflow paid by the contingent claim and the solution to the following equation [see Cox, Ingersoll and Ross (1985a)]:

$$\tfrac{1}{2}\sigma^2 F_{CC} + \left(\alpha C - \lambda_C\right) F_C + \tfrac{1}{2}\sigma_r^2 r F_{rr} + \left[\kappa\theta - \left(\kappa + \lambda_r\right) r\right] F_r - r\left(1 - \tau_p\right) F = 0 \tag{1A}$$

Since cash flow and interest rates processes are orthogonal, the solution to (1A) can be expressed in terms of a product of independent functions of separate arguments, cash flow and interest rates. By separation of variables, we can derive the following system of ordinary differential equation:

$$\tfrac{1}{2}\sigma^2 G_{CC} + \left(\alpha C - \lambda_C\right) G_C - K_2 G = 0 \tag{2A}$$

$$\tfrac{1}{2}\sigma_r^2 r H_{rr} + \left[\kappa\theta - \left(\kappa + \lambda_r\right) r\right] H_r + \left[K_2 - r\left(1 - \tau_p\right)\right] H = 0 \tag{3A}$$

Having divided both sides of the equation (2A) by ½σ^2, we apply following transformation:

$$\frac{2\alpha}{\sigma^2} C - 2\lambda_C = \sqrt{-\frac{4\alpha}{\sigma^2} Y}$$

Hence, the equation (2A) will be transformed to Weiler's canonical form (Kummer equation) [see Abromowitz and Stegun (1972), Bateman (1918), and Zwillinger (1998)]:

$$Y G_{YY} + \left(\frac{1}{2} - Y\right) G_Y + \frac{K_2}{2\alpha} Y = 0$$

The general solution to the aforementioned equation (Kummer Equation) is as follows:

$$G(Y) = \hat{K}\, M\left(-\frac{K_1}{2\alpha}, \frac{1}{2}; Y\right)$$

and thus, by substituting for *Y*, we have:

$$G(C) = \hat{K}\, M\left(Q^C(K_2), \tfrac{1}{2}; \psi(C)\right)$$

Where;

$$Q^C(K_2) \equiv -\frac{K_2}{2\alpha}$$

$$\psi(C) \equiv -\frac{\alpha}{\sigma^2}\left(C - \frac{\lambda_C \sigma^2}{\alpha}\right)^2$$

The equation (3A) can be transformed to Weiler's canonical form (Kummer equation) [see Abromowitz and Stegun (1972), Bateman (1918), and Zwillinger (1998)]. Let's assume that:

$$\begin{aligned} H(r) &= e^{\varphi r} Z(X) \\ X &= \varsigma r \end{aligned} \tag{4A}$$

After some algebra, the equation (3A) then can be written as:

$$\begin{aligned} &X Z'' + \left[\frac{2\kappa\theta}{\sigma_r^2 \varsigma} + 2\left(\varphi - \frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2}\right)\frac{X}{\varsigma}\right] Z' \\ &\quad + \left[\frac{2\kappa\theta\varphi}{\sigma_r^2\, \varsigma^2} + \frac{2K_2}{\sigma_r^2\, \varsigma^2} + \left(\varphi^2 - 2\frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2}\varphi - \frac{2(1-\tau_p)}{\sigma_r^2}\right)\frac{X}{\varsigma^2}\right] Z = 0 \end{aligned} \tag{5A}$$

Then, we choose ϕ and η such that:

$$\begin{gathered} 2\left(\varphi - \frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2}\right)\frac{1}{\varsigma} = -1 \\ \varphi^2 - 2\frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2}\varphi - \frac{2(1-\tau_p)}{\sigma_r^2} = 0 \end{gathered}$$

Having solved the aforementioned equations, we have:

$$X\, Z'' + \left(S^r - X\right) Z' - Q^r(K_2)\, Z = 0 \tag{6A}$$

Where;

$$\begin{aligned} S^r &\equiv \frac{2\kappa\theta}{\sigma_r^2 \varsigma} \\ Q^r(K_2) &\equiv -\frac{\varphi}{\varsigma} S - \frac{2K_2}{\sigma_r^2 \varsigma^2} \\ \varphi &\equiv \frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2} - \frac{1}{2}\varsigma \\ \varsigma &\equiv 2\left[\left(\frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2}\right)^2 + \frac{2(1-\tau_p)}{\sigma_r^2}\right]^{\frac{1}{2}} \end{aligned}$$

The solution to equation (5A) (i.e. Kummer equation) is given by:

$$Z(X) = \widehat{K}\, M(Q^r(K_2), S; X)$$

where, *M(a,b;z)* is the Kummer function. Applying the relevant inverse transformation, we then arrive at:

$$H(r) = \widehat{K}\, M(Q^r(K_2), S; \varsigma r)\, e^{\varphi r}$$

Hence, substituting for functions *G*(*C*) and *H*(*r*), we find that the functional form of the *F*(*C*,*r*;***X***) is as defined by (17).

.∎

A.2. Proof of Lemma 1.

The solution to equation (26) is consist of the present value of continuous cash outflow paid by the contingent claim and the solution to the following equation [see Cox, Ingersoll and Ross (1985a)]:

$$\begin{aligned} &\tfrac{1}{2}\sigma^2 F_{CC} + \left(\alpha C - \lambda_C \sigma^2\right) F_C + \tfrac{1}{2}\kappa^2 F_{rr} - \kappa w F_w - \tfrac{1}{\kappa}\left(1-\tau_p\right) w F \\ &\quad + \tfrac{1}{2}\zeta^2 \theta F_{\theta\theta} + \left[\xi\overline{\theta} - \left(\xi + \lambda_\theta \zeta^2\right)\theta\right] F_\theta - \left(1-\tau_p\right)\theta F \\ &\quad + \tfrac{1}{2}\eta^2 \upsilon F_{\upsilon\upsilon} + \left[\varpi\overline{\upsilon} - \left(\varpi + \lambda_\upsilon \eta^2\right)\upsilon\right] F_\upsilon - \tfrac{1}{\kappa}\left(1-\tau_p\right)\lambda_r \upsilon F = 0 \end{aligned} \tag{7A}$$

Since cash flow and term structure processes are orthogonal, the solution to (1A) can be expressed in terms of a product of independent functions of separate arguments, cash flow, instantaneous rate, long-rates and volatility. By separation of variables, we can derived the following system of ordinary differential equation:

$$\tfrac{1}{2}\sigma^2 G_{CC} + \left(\alpha C - \lambda_C \sigma^2\right) G_C - K_2 G = 0 \tag{8A}$$

$$\tfrac{1}{2}\kappa^2 H_{ww} - \kappa w H_w + \left[K_2 - \tfrac{1}{\kappa}\left(1-\tau_p\right) w\right] H = 0 \tag{9A}$$

$$\tfrac{1}{2}\zeta^2 \theta M_{\theta\theta} + \left[\xi\overline{\theta} - \left(\xi - \lambda_\theta \zeta^2\right)\theta\right] M_\theta - \left(1-\tau_p\right)\theta M = 0 \tag{10A}$$

$$\tfrac{1}{2}\eta^2 \upsilon N_{\upsilon\upsilon} + \left[\varpi\overline{\upsilon} - \left(\varpi - \lambda_\upsilon \eta^2\right)\upsilon\right] N_\upsilon - \tfrac{1}{\kappa}\left(1-\tau_p\right)\lambda_r \upsilon N = 0 \tag{11A}$$

Similar to the one-factor case, the solution to *G*(*C*) is as follow:

$$G(C) = M\left(Q^C(K_2), \tfrac{1}{2}; \psi(C)\right)$$

Where;

$$Q^C(K_2) \equiv -\frac{K_2}{2\alpha}$$

$$\psi(C) \equiv -\frac{\alpha}{\sigma^2}\left(C - \frac{\lambda_C \sigma^2}{\alpha}\right)^2$$

The equation (9A) can be transformed to Weiler's canonical form (Kummer equation) [see Abromowitz and Stegun (1972), Bateman (1918), and Zwillinger (1998)]. Let's assume that:

$$H(w) = e^{-\frac{(1-\tau_p)w}{\kappa}} X(w)$$

The equation (3C) then can be rewritten as following:

$$X'' + (h + f\,r)X' + g\,X = 0$$

where;

$$h \equiv -\frac{2(1-\tau_p)}{\kappa^2}$$
$$f \equiv -\frac{2}{\kappa}$$
$$g \equiv \frac{1}{\kappa^2}\left[\left(\frac{1-\tau_p}{\kappa}\right)^2 + 2K_2\right]$$

Let's assume that $\varsigma \equiv -(f + g\,r)^2 / 2g$, then, the aforementioned equation can be transformed into canonical Weiler's form such as:

$$\varsigma\,X'' + \left(\tfrac{1}{2} - \varsigma\right)X' - Q^r\,X = 0$$

where;

$$Q^w \equiv -\frac{K_2}{2\kappa} - \frac{(1-\tau_p)^2}{4\kappa^3}$$

Substituting for *y* and *X*, we derive:

$$H(w) = \hat{K}_2\,e^{-\frac{(1-\tau_p)\,w}{\kappa}} M\left(Q^w, \tfrac{1}{2}; \varsigma(w)\right)$$

where, *M(a,b;z)* is the Kummer function.

Since equations (10A) and (11A) are structurally identical, we solve for (11A) and hence apply the general solutions for (11A), correcting for parametric differences. To transform (10A) to canonical form of Kummer equation, we follow same methodology as we did in equation (3A), thus:

$$M(\theta) = \hat{K}_3 \, M(Q^{\theta}(K_3), S^{\theta}; \varsigma^{\theta}\theta)\, e^{\varphi^{\theta}\theta}$$

where;

$$\begin{aligned}
Q^{\theta} &\equiv -\frac{\varphi^{\theta}}{\varsigma^{\theta}} S^{\theta} \\
S^{\theta} &\equiv \frac{2\xi\overline{\theta}}{\zeta^{2}\varsigma^{\theta}} \\
\varphi^{\theta} &\equiv \frac{\xi + \lambda_{\theta}\zeta^{2}}{\zeta^{2}} - \frac{1}{2}\varsigma^{\theta} \\
\varsigma^{\theta} &\equiv 2\left[\left(\frac{\xi + \lambda_{\theta}\zeta^{2}}{\zeta^{2}}\right)^{2} + \frac{2(1-\tau_{p})}{\zeta^{2}}\right]^{\frac{1}{2}}
\end{aligned}$$

Using same methodology, we derive $N(\upsilon)$ is:

$$N(\upsilon) = \hat{K}_4 \, M(Q^{\upsilon}, S^{\upsilon}; \varsigma^{\upsilon}\upsilon)\, e^{\varphi^{\upsilon}\upsilon}$$

where;

$$\begin{aligned}
Q^{\upsilon} &\equiv -\frac{\varphi^{\upsilon}}{\varsigma^{\upsilon}} S^{\upsilon} \\
S^{\upsilon} &\equiv \frac{2\varpi\overline{\upsilon}}{\eta^{2}\varsigma^{\upsilon}} \\
\varphi^{\upsilon} &\equiv \frac{\varpi + \lambda_{\upsilon}\eta^{2}}{\eta^{2}} - \frac{1}{2}\varsigma^{\upsilon} \\
\varsigma^{\upsilon} &\equiv 2\left[\left(\frac{\varpi + \lambda_{\upsilon}\eta^{2}}{\eta^{2}}\right)^{2} + \frac{2\frac{1}{\kappa}(1-\tau_{p})\lambda_{r}}{\eta^{2}}\right]^{\frac{1}{2}}
\end{aligned}$$

Substituting for $G(C)$, $H(r)$, $M(\theta)$, and $N(\upsilon)$, we have:

$$\begin{aligned}
\hat{G}(\omega,\psi,\varphi,\eta;C,w,\theta,\upsilon;\boldsymbol{X}) = K_1(\omega,\psi,\varphi,\eta)\, & M\left(Q^{C}(K_2(\omega,\psi,\varphi,\eta)), \tfrac{1}{2}; \psi(C)\right) \\
& \times M\left(Q^{w}(K_2(\omega,\psi,\varphi,\eta)), \tfrac{1}{2}; \varsigma(w)\right) \\
& \times M(Q^{\theta}, S^{\theta}; \varsigma^{\theta}\theta)\, M(Q^{\upsilon}, S^{\upsilon}; \varsigma^{\upsilon}\upsilon) \\
& \times \exp\left\{(1-\tau_{p})\left[(r-\theta)\upsilon^{-\frac{1}{2}} + \lambda_{r}\upsilon\right] + \varphi^{\theta}\theta + \varphi^{\upsilon}\upsilon\right\}
\end{aligned}$$

.∎

A.3. Proof of Proposition 2.

Using methods of perturbation, the solution to equation (29) can be written as:

$$F(C,w,\theta,\upsilon) = F_0(C,w,\theta,\upsilon) + F_1(C,w,\theta,\upsilon) + F_2(C,w,\theta,\upsilon) + \cdots \tag{32}$$

where;

$$F(C,w,\theta,\upsilon) = \int\int\int\int G(C,w,\theta,\upsilon;\hat{C},\hat{w},\hat{\theta},\hat{\upsilon})\left[\delta + \boldsymbol{V}\left[\hat{C},\hat{w},\hat{\theta},\hat{\upsilon}\right]F(\hat{C},\hat{w},\hat{\theta},\hat{\upsilon})\right]d\hat{C}\,d\hat{w}\,d\hat{\theta}\,d\hat{\upsilon}$$

Expanding the above integral, we arrive at (32).

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A.4. Proof of Proposition 3.

Using methods of separation of variables and splitting equations, the solution to equation (29) should satisfy the following system of ordinary differential equations:

$$\begin{aligned} &\tfrac{1}{2}\sigma^2\pi\, F_{CC} + \left(\alpha C - \lambda_C\sigma^2\right)\pi F_C + \tfrac{1}{2}\sigma_r^2 rF_{rr} + \left[\kappa\theta - (\kappa+\lambda_r)r\right]F_r \\ &\quad + \tfrac{1}{2}\sigma_\pi^2\pi^3 F_{\pi\pi} + \left[\left(\kappa_\pi\bar{\pi} - \lambda_\pi^1\sigma_\pi^2\right) - \left(\kappa_\pi + \lambda_\pi^1\right)\pi\right]\pi F_r - r\left(1-\tau_p\right)F + \delta(t) = 0 \end{aligned} \tag{29}$$

Since cash flow and interest rates processes are orthogonal, the solution to (1A) can be expressed in terms of a product of independent functions of separate arguments, cash flow and interest rates. By separation of variables, we can derive the following system of ordinary differential equation:

$$\tfrac{1}{2}\sigma^2 G_{CC} + \left(\alpha C - \lambda_C\right)G_C - K_2 G = 0 \tag{12A}$$

$$\tfrac{1}{2}\sigma_r^2\, rH_{rr} + \left[\kappa\theta - (\kappa+\lambda_r)r\right]H_r - r\left(1-\tau_p\right)H = 0 \tag{13A}$$

$$\tfrac{1}{2}\sigma_\pi^2\,\pi^3 W_{\pi\pi} + \left[\left(\kappa_\pi\bar{\pi} - \lambda_{\pi,1}\sigma_\pi^2\right) - \left(\kappa_\pi + \lambda_{\pi,2}\right)\pi\right]\pi W_\pi + K_2\pi W = 0 \tag{14A}$$

The solutions to equation (12A) and (13A) are similar to those of (2A) and (3A), hence:

$$G(C) = \hat{K}\, M\left(Q^C(K_2), \tfrac{1}{2}; \psi(C)\right)$$

Where;

$$Q^C(K_2) \equiv -\frac{K_2}{2\alpha}$$

$$\psi(C) \equiv -\frac{\alpha}{\sigma^2}\left(C - \frac{\lambda_C \sigma^2}{\alpha}\right)^2$$

and;

$$H(r) = \hat{\hat{K}}\, M(Q^r, S; \varsigma r)\, e^{\varphi r}$$

where;

$$S^r \equiv \frac{2\kappa\theta}{\sigma_r^2 \varsigma}$$
$$Q^r(K_2) \equiv -\frac{\varphi}{\varsigma} S$$
$$\varphi \equiv \frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2} - \frac{1}{2}\varsigma$$
$$\varsigma \equiv 2\left[\left(\frac{\kappa + \lambda_r \sigma_r^2}{\sigma_r^2}\right)^2 + \frac{2(1-\tau_p)}{\sigma_r^2}\right]^{\frac{1}{2}}$$

For equation (14A) similar to Lemma 1, we postulate a solution of the form:

$$H(r) = \widehat{K}\, y^{\eta}\ f(y); \qquad y \equiv \frac{\Delta}{\pi}$$

Taking appropriate derivatives and substituting in the equation, we can show that the solution to equation (14A) should solve following ODE:

$$\tfrac{1}{2}\sigma_\pi^2\, y^2\, f'' + \left[\left(\kappa*_\pi + \sigma_\pi^2(\eta+1)\right) - \frac{\pi*}{\Delta} y\right] f' + \left[-\frac{\eta\pi*}{\Delta} y + K_2 + \kappa*_\pi\, \eta + \tfrac{1}{2}\sigma_\pi^2 \eta(\eta+1)\right] f = 0 \tag{15A}$$

where;

$$\pi* = \kappa_\pi \bar{\pi} - \lambda_{\pi,1}\sigma_\pi^2$$
$$\kappa*_\pi = \kappa_\pi + \lambda_{\pi,2}$$

It can be easily shown that if the following conditions is met:

$$K_2 + \kappa*_\pi\, \eta + \tfrac{1}{2}\sigma_\pi^2 \eta(\eta+1) = 0$$

Solving for the η, we have:

$$\eta \equiv -\left(\frac{\kappa *_{\pi}}{\sigma_{\pi}^{2}} + \frac{1}{2}\right) - \left[\left(\frac{\kappa *_{\pi}}{\sigma_{\pi}^{2}} + \frac{1}{2}\right)^{2} - \frac{2K_{2}}{\sigma_{\pi}^{2}}\right]^{\frac{1}{2}}$$

Then equation (15A) can be transformed to Weiler's canonical form (Kummer equation) [see Abromowitz and Stegun (1972), Bateman (1918), and Zwillinger (1998)]. Thus, the *f*(*y*) solves:

$$y\, f'' + \left[\left(\frac{2\kappa *_{\pi}}{\sigma_{\pi}^{2}} + 2(\eta + 1)\right) - \frac{2\pi *}{\Delta\sigma_{\pi}^{2}}\, y\right] f' - \frac{2\eta\pi *}{\Delta\sigma_{\pi}^{2}}\, f = 0 \tag{16A}$$

The aforementioned is a Kummer function for which there exists a solution of the form:

$$f(y) = M(Q^{\pi}, S^{\pi}; y)$$

where;

$$Q^{\pi} \equiv \eta$$

$$S^{\pi} \equiv \frac{2\kappa *_{\pi}}{\sigma_{\pi}^{2}} + 2(\eta + 1)$$

$$\eta \equiv -\left(\frac{\kappa *_{\pi}}{\sigma_{\pi}^{2}} + \frac{1}{2}\right) - \left[\left(\frac{\kappa *_{\pi}}{\sigma_{\pi}^{2}} + \frac{1}{2}\right)^{2} - \frac{2K_{2}}{\sigma_{\pi}^{2}}\right]^{\frac{1}{2}}$$

$$y \equiv \frac{\Delta}{\pi}$$

and;

$$\Delta \equiv \frac{\sigma_{\pi}^{2}}{2\pi *}$$

Hence, substituting for functions *G*(*C*), *H*(*r*) and *W*(*π*), we find that the functional form of the *F*(*C*,*r*;***X***) is as defined by (17).

.▮

A.5. Proof of Lemma 2.

We postulate that solution to fundamental valuation equation (36) is of the following form:

$$P(r,T) = A(T)\exp\{-B(T)\, r\} \qquad ()$$

Thus using Taylor's series expansion, equation (36) can be written as:

$$\tfrac{1}{2}\sigma_{r}^{2}\, r\, P_{rr} + [\kappa\theta - (\kappa + \lambda_{r})\, r]\, P_{r} + \left[h\left(\tfrac{1}{2}(\mu_{r}^{2} + \gamma_{r}^{2})\, B^{2} - \mu_{r}\, B\right) - r\right] P + P_{T} \equiv 0$$

Using method of separation of variables, the solution to aforementioned PDE has to solve following system of ordinary differential equation:

$$B' - \tfrac{1}{2}\sigma_r^2 B^2 - \left(\kappa + \lambda_r\right)B + 1 \equiv 0 \tag{20A}$$

$$\frac{A'}{A} + \tfrac{1}{2} h(\mu_r^2 + \gamma_r^2)\, B^2 - \left(\kappa\theta + h\mu_r\right)B \equiv 0 \tag{21A}$$

The solution to equation (20A) is same as CIR's, hence:

$$B(T) \equiv \frac{2\gamma\, e^{\frac{1}{2}\gamma T}}{\left(\kappa + \lambda *_r + \gamma\right)\left(e^{\gamma T} - 1\right) + 2\gamma}$$

where;

$$\gamma \equiv \left[\left(\kappa + \lambda *_r\right)^2 + 2\sigma_r^2\right]^{\frac{1}{2}}$$

The solution to equation (21A) is:

$$A(T) = -\tfrac{1}{2} h(\mu_r^2 + \gamma_r^2) \int_0^T B(\tau)^2\, d\tau \quad + \left(\kappa\theta + h\mu_r\right) \int_0^T B(\tau)\, d\tau$$

Integrating over appropriate domain, it yields:

$$\begin{aligned} A(T) \equiv \exp&\left\{ \frac{4\Delta\left(e^{\gamma T} - 1\right)}{\sigma_r^2\left[\left(\kappa + \lambda *_r + \gamma\right)\left(e^{\gamma T} - 1\right) + 2\gamma\right]} + \frac{2\left(\gamma - \kappa - \lambda *_r\right)\Omega - 4\gamma^2\Delta}{\left(\gamma - \kappa - \lambda *_r\right)^2} T \right\} \times \\ &\left[\frac{1}{2}\left(\frac{\kappa + \lambda *_r}{\gamma} + 1\right)\left(e^{\gamma T} - 1\right) + 1\right]^{\frac{4(\kappa + \lambda *_r)\Delta - 2\sigma_r^4\Omega}{\sigma_r^4}} \end{aligned} \tag{22A}$$

where;

$$\Delta \equiv \tfrac{1}{2} h_r \left(\mu_r^2 + \gamma_r^2\right)$$
$$\Omega \equiv -\left(\kappa\,\theta + h_r\, \mu_r\right)$$

.∎

A.6. Proof of Lemma 3.

Having applied transformations $\hat{C} \equiv C + \mu_C$ and $\tilde{r} \equiv r + \mu_r + \sigma_r^2 / 2h_r$, I then use methods of separation of variables to splitting equation (43) into following ordinary differential equations:

$$\left(\hat{C}^2+\Delta^2\right)U''+\left(\hat{\alpha}\,\hat{C}-\hat{\mu}\right)U'-K^*U=0 \tag{23A}$$

$$\left(\hat{r}^2+\Omega^2\right)W''+\left(\Theta-\Lambda\,\hat{r}\right)W'-\left(\Psi-\Phi\,\hat{r}\right)W=0 \tag{24A}$$

where, the parameters are defined as following:

$$\nu\equiv\tfrac{1}{2}\left\lfloor\left(\hat{\alpha}-2\,\hat{\alpha}+4\,K^*+1\right)^{\frac{1}{2}}-1\right\rfloor$$

$$\mu\equiv\tfrac{1}{2}\left[\left(\hat{\alpha}-2-i\,\frac{\hat{\mu}}{2\,\Delta}\right)^2-\frac{3\,\hat{\mu}^2}{2\,\hat{\Delta}^2}\right]^{\frac{1}{2}}$$

$$\Delta\equiv\left[\gamma_C^2+\frac{\sigma^2}{h_C}\right]^{\frac{1}{2}}$$

$$\hat{\alpha}\equiv\frac{2\,\alpha}{h_C}+2$$

$$\hat{\mu}\equiv\frac{2}{h_C}\left(\alpha\,\mu_C+\lambda_C\,\sigma^2\right)$$

$$K^*\equiv\frac{2\,K_2}{h_C}$$

$$\Omega\equiv\gamma_r^2-\frac{\sigma_r^2}{h_r}\left(\mu_r+\frac{\sigma_r^2}{4\,h_r}\right)$$

$$\Theta\equiv\left[\kappa\,\theta-\sigma_r^2+\left(\kappa+\lambda_r\,\sigma_r^2\right)\left(\mu_r+\frac{\sigma_r^2}{h_r}\right)\right]\frac{2}{h_r}$$

$$\Lambda\equiv\left(\kappa+\lambda_r\,\sigma_r^2-h_r\right)\frac{2}{h_r}$$

$$\Psi\equiv\frac{2}{h_r}\left[\left(1-\tau_p\right)\left(\mu_r+\frac{\sigma_r^2}{2\,h_r}\right)+K_2\right]$$

$$\Phi\equiv\frac{2\left(1-\tau_p\right)}{h_r}$$

In equation (23A), I posit that the solution is of the following form:

$$\Gamma(y)\equiv\left(1-y^2\right)^{\eta}\,f(y)\exp\left(-\phi\arctan(i\,y)\right)$$

where, $y\equiv-i\,C/\Delta$ thus, after tedious algebra, we have:

$$\left(1-y^2\right)f''-2\,y\,f'+\left[\nu(\nu+1)-\frac{\mu^2}{1-y^2}\right]f=0 \tag{25A}$$

where, the parameters are defined by (46). The general solution to the equation (25A) is a Legendre function (see Abramowitz and Stegun (1999)), which can be expressed as:

$$f(y) = P_{\nu}^{\mu}(y) = \frac{1}{\Gamma(1-\mu)} \left[\frac{y+1}{y-1} \right]^{\frac{1}{2}\mu} F\left(-\nu, \nu+1; 1-\mu; \frac{1-y}{2} \right)$$

in which, Γ(.) is the Gamma fucntion, and $F(a,b;c;z)$ is the hypergoemetric function and can be expressed in an infinite series form such as following:

$$F\left(a, b; c; z\right) \equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n}$$

For equation (24A), a general solution does not exists. However, like the three-factor model, one can use the Green theorem and method of perturbation to derive an approximate solution form for (24A).

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A.6. Proof of Proposition 4.

Like proposition 3, we perturb an approximate solution using Green's function. See details in Zwillinger (1996).

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Appendix B.

B.1. Case of One-Factor Term Structure:

B.1.1. Net Benefit's Coefficients.

A solution to (21) has to satisfy the boundary conditions of (19), or:

$$\begin{aligned} NB(C^U, r^U; \boldsymbol{X}) &= \rho^U NB(C^U, r^U; \boldsymbol{X}) - \kappa_F - \rho^U \kappa_V B \\ NB(C^L, r^L; \boldsymbol{X}) &= \rho^L NB(C^L, r^L; \boldsymbol{X}) - \kappa_F - \rho^L \kappa_V B - G - g B \end{aligned} \tag{1B}$$

Substituting for C and r in appropriate equations, $\boldsymbol{A} \equiv (A_1, A_2)$ is a solution for the following nonlinear system of equations:

$$\begin{aligned} & A_1 \Big[M\left(Q^C(A_2), \tfrac{1}{2}; \psi(C^U)\right) M\left(Q^r(A_2), S^r; \varsigma r^U\right) e^{\varphi r^U} \\ & \quad - \rho^U M\left(Q^C(A_2), \tfrac{1}{2}; \psi(C_0)\right) M(Q^r(A_2), S^r; \varsigma r_0) e^{\varphi r_0} \Big] \\ & \quad = \delta \left(\rho^U \Pi(r_0) - \Pi(r^U)\right) - \kappa_F - \rho^U \kappa_V B \\ & A_1 \Big[M\left(Q^C(A_2), \tfrac{1}{2}; \psi(C^L)\right) M\left(Q^r(A_2), S^r; \varsigma r^L\right) e^{\varphi r^L} \\ & \quad - \rho^L M\left(Q^C(A_2), \tfrac{1}{2}; \psi(C_0)\right) M(Q^r(A_2), S^r; \varsigma r_0) e^{\varphi r_0} \Big] \\ & \quad = \delta \left(\rho^L \Pi(r_0) - \Pi(r^L)\right) - \kappa_F - \rho^L \kappa_V B - G - gB \end{aligned} \tag{2B}$$

and;

$$\delta = \left(\tau_c - \tau_p\right)\iota - \left(1 - \tau_p\right)\kappa_m mB - \kappa_F - \kappa_V B$$

$$\Pi\left(r\right) \equiv \int_0^\infty P\left(r,\tau\right)d\tau$$

Though solving systems of nonlinear equations is one of the more challenging numerical methods, I this case one can easily disentangle the aforementioned systems of equations to two separately identifiable nonlinear equations. Let's assume that

$$\Theta(C,r;\boldsymbol{X}) \equiv M\left(Q^C(A_2),\tfrac{1}{2};\psi(C)\right)M\left(Q^r(A_2),S^r;\varsigma\, r\right)e^{\varphi r}$$

thus by dividing the two equations of (2B) by each other we have:

$$\frac{\Theta(C^U,r^U;\boldsymbol{X}) - \rho^U\Theta(C_0,r_0;\boldsymbol{X})}{\Theta(C^L,r^L;\boldsymbol{X}) - \rho^L\Theta(C_0,r_0;\boldsymbol{X})} = \lambda_{NB} \tag{3B}$$

where,

$$\lambda_{NB} = \frac{\delta\left(\Pi\left(r^U\right) - \rho^U\Pi\left(r_0\right)\right) + \kappa_F + \rho^U\kappa_V B}{\delta\left(\Pi\left(r^L\right) - \rho^L\Pi\left(r_0\right)\right) + \kappa_F + \rho^L\kappa_V B + G + gB}$$

Since Kummer function can also be written in an integral form as:

$$M(Q,S,z) = \frac{\Gamma(S)}{\Gamma(S-Q)\Gamma(Q)}\int_0^1 e^{zt}\,t^{Q-1}(1-t)^{S-Q-1}\,dt \quad = \frac{\Gamma(S)}{\Gamma(S-Q)\Gamma(Q)} \times m(Q,S,z)$$

where, Γ(.) is the Gamma function, and *m*(*Q*,*S*,*z*) is defined as:

$$m(Q,S,z) \equiv \int_0^1 e^{zt}\,t^{Q-1}(1-t)^{S-Q-1}\,dt$$

Thus, after some algebra, it can be shown that the equation (3B) can be written as following:

$$\begin{aligned} & m\left(Q^C(A_2),\tfrac{1}{2};\psi(C^U)\right)m\left(Q^r(A_2),S^r;\varsigma\, r^U\right)e^{\varphi r^U} \\ & \quad - \lambda_{NB}\, m\left(Q^C(A_2),\tfrac{1}{2};\psi(C^L)\right)m\left(Q^r(A_2),S^r;\varsigma\, r^L\right)e^{\varphi r^L} \\ & \quad - \left(\rho^U - \lambda_{NB}\,\rho^U\right)m\left(Q^C(A_2),\tfrac{1}{2};\psi(C_0)\right)m(Q^r(A_2),S^r;\varsigma r_0)e^{\varphi r_0}\Big] \equiv 0 \end{aligned} \tag{4B}$$

Hence, equation (4B) and either equations in (2B) solve for $\boldsymbol{A} \equiv (A_1 , A_2)$. Though there are number of methods that can be implemented to solve (4B), we exploit Some of the characteristics of the Kummer function to solve the aforementioned equations more efficiently.

Since the Kummer function and $m(Q,S,z)$ have integral representation, having rearranged the integrands in (4B), we can write:

$$\int_0^1 \int_0^1 \left[e^{\psi(C^U)t+r^U(\tau+\varphi)} - \lambda e^{\psi(C^L)t+r^L(\tau+\varphi)} - \left(\rho^U - \lambda \rho^L\right) e^{\psi(C_0)t+r_0(\tau+\varphi)}\right] \times \left(1-\tfrac{1}{t}\right)^{1-Q_C} \left(1-\tfrac{1}{\tau}\right)^{1-Q_r} \left(1-t\right)^{-\frac{3}{2}} \left(1-\tau\right)^{S_r-2} \quad dt\, d\tau \equiv 0$$

B.1.2. Debt's Coefficients.

A solution to (3) has to satisfy the boundary conditions of (11) and (12), or:

$$\begin{aligned} D(C^U, r^U; \boldsymbol{X}) &= \left(1+\beta\right) B \\ D(C^L, r^L; \boldsymbol{X}) &= \left(1-g\right) B - G \end{aligned} \tag{5B}$$

Substituting for C and r in appropriate equations, $\boldsymbol{A^*} \equiv (A^*_1 , A^*_2)$ is a solution for the following nonlinear system of equations:

$$\begin{aligned} &A^*_1 \left[M\left(Q^C(A^*_2), \tfrac{1}{2}; \psi(C^U)\right) M\left(Q^r(A^*_2), S^r; \varsigma r^U\right) e^{\varphi r^U}\right. \\ &\qquad = -\left(\iota + mB\right)\Pi\left(r^U\right) + \left(1+\beta\right)B \\ &A^*_1 \left[M\left(Q^C(A^*_2), \tfrac{1}{2}; \psi(C^L)\right) M\left(Q^r(A^*_2), S^r; \varsigma r^L\right) e^{\varphi r^L}\right. \\ &\qquad = -\left(\iota + mB\right)\Pi\left(r^L\right) + \left(1-g\right)B - G \end{aligned}$$

and;

$$\delta = \left(1-\tau_p\right)\left(\iota + mB\right)$$

$$\Pi(r) \equiv \int_0^\infty P(r,\tau)\, d\tau$$

Similar to the net benefit's case, it can be shown that solving for A^*_2 in the aforementioned system of nonlinear equations is equivalent to solving the following equation:

$$\begin{aligned} &m\left(Q^C(A^*_2), \tfrac{1}{2}; \psi(C^U)\right) m\left(Q^r(A^*_2), S^r; \varsigma r^U\right) e^{\varphi r^U} \\ &\quad - \lambda_{DB}\, m\left(Q^C(A^*_2), \tfrac{1}{2}; \psi(C^L)\right) m\left(Q^r(A^*_2), S^r; \varsigma r^L\right) e^{\varphi r^L} \equiv 0 \end{aligned} \tag{6B}$$

where,

$$\lambda_{DB}=\frac{-(\iota+mB)\Pi\left(r^{U}\right)+(1+\beta)B}{-(\iota+mB)\Pi\left(r^{L}\right)+(1-g)B-G}$$

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B.2. Case of Three-Factor Term Structure:

B2.1. Net Benefit's Coefficients.

A solution to (21) has to satisfy the boundary conditions of (19), or:

$$\begin{aligned}
&\hat{G}(\omega,\mathbf{\Omega};C^{U},\boldsymbol{R}^{U};\boldsymbol{X})=\rho^{U}\hat{G}(\omega,\mathbf{\Omega};C^{U},\boldsymbol{R}^{U};\boldsymbol{X})-\left(\rho^{U}\kappa_{V}B+\kappa_{F}\right)e^{iC^{U}\omega+i\,\mathbf{\Omega}\boldsymbol{R}^{U}}\\
&\hat{G}(\omega,\mathbf{\Omega};C^{L},\boldsymbol{R}^{L};\boldsymbol{X})=\rho^{L}\hat{G}(\omega,\mathbf{\Omega};C^{L},\boldsymbol{R}^{L};\boldsymbol{X})-\left(\rho^{L}\kappa_{V}B+\kappa_{F}+gB+G\right)e^{iC^{U}\omega+i\,\mathbf{\Omega}\boldsymbol{R}^{L}}
\end{aligned}\tag{7B}$$

where $\boldsymbol{R^*}\equiv(w,\theta,\upsilon)^{\mathsf{T}}$, $\boldsymbol{R^*}^{U}\equiv(w^{U},\theta^{U},\upsilon^{U})^{\mathsf{T}}$, $\boldsymbol{R^*}^{L}\equiv(w^{L},\theta^{L},\upsilon^{L})^{\mathsf{T}}$, and $\mathbf{\Omega}\equiv(\psi,\varphi,\eta)$. Substituting for C and $\boldsymbol{R^*}$ in appropriate equations, $\boldsymbol{a}\equiv(a_1\,,\,a_2)$ is a solution for the following nonlinear system of equations:

$$\begin{aligned}
&a_1(\omega,\mathbf{\Omega})\Bigg\{M\left(Q^{C}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\psi(C^{U})\right)M\left(Q^{w}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\varsigma(w^{U})\right)\\
&\qquad\times M\left(Q^{\theta},S^{\theta};\varsigma^{\theta}\theta^{U}\right)M\left(Q^{\upsilon},S^{\upsilon};\varsigma^{\upsilon}\upsilon^{U}\right)exp\left\{\left(1-\tau_{p}\right)\left[\left(r^{U}-\theta^{U}\right)\upsilon^{U^{-\frac{1}{2}}}+\lambda_{r}\upsilon^{U}\right]+\varphi^{\theta}\theta^{U}+\varphi^{\upsilon}\upsilon^{U}\right\}\\
&-\rho^{U}\Bigg[M\left(Q^{C}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\psi(C_0)\right)M\left(Q^{w}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\varsigma(w_0)\right)\\
&\qquad\times M\left(Q^{\theta},S^{\theta};\varsigma^{\theta}\theta_0\right)M\left(Q^{\upsilon},S^{\upsilon};\varsigma^{\upsilon}\upsilon_0\right)exp\left\{\left(1-\tau_{p}\right)\left[\left(r_0-\theta_0\right)\upsilon_0^{-\frac{1}{2}}+\lambda_{r}\upsilon_0\right]+\varphi^{\theta}\theta_0+\varphi^{\upsilon}\upsilon_0\right\}\Bigg]\Bigg\}\\
&=-\left(\rho^{U}\kappa_{V}B+\kappa_{F}\right)e^{iC^{U}\omega+i\,\mathbf{\Omega}\boldsymbol{R}^{U}}
\end{aligned}$$

$$\begin{aligned}
&a_1(\omega,\mathbf{\Omega})\Bigg\{M\left(Q^{C}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\psi(C^{L})\right)M\left(Q^{w}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\varsigma(w^{L})\right)\\
&\qquad\times M\left(Q^{\theta},S^{\theta};\varsigma^{\theta}\theta^{L}\right)M\left(Q^{\upsilon},S^{\upsilon};\varsigma^{\upsilon}\upsilon^{L}\right)exp\left\{\left(1-\tau_{p}\right)\left[\left(r^{L}-\theta^{L}\right)\upsilon^{L^{-\frac{1}{2}}}+\lambda_{r}\upsilon^{L}\right]+\varphi^{\theta}\theta^{L}+\varphi^{\upsilon}\upsilon^{L}\right\}\\
&-\rho^{L}\Bigg[M\left(Q^{C}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\psi(C_0)\right)M\left(Q^{w}(a_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\varsigma(w_0)\right)\\
&\qquad\times M\left(Q^{\theta},S^{\theta};\varsigma^{\theta}\theta_0\right)M\left(Q^{\upsilon},S^{\upsilon};\varsigma^{\upsilon}\upsilon_0\right)exp\left\{\left(1-\tau_{p}\right)\left[\left(r_0-\theta_0\right)\upsilon_0^{-\frac{1}{2}}+\lambda_{r}\upsilon_0\right]+\varphi^{\theta}\theta_0+\varphi^{\upsilon}\upsilon_0\right\}\Bigg]\Bigg\}\\
&=-\left(\rho^{L}\kappa_{V}B+\kappa_{F}+gB+G\right)e^{iC^{L}\omega+i\,\mathbf{\Omega}\boldsymbol{R}^{L}}
\end{aligned}$$

Similar to the net benefit's case, it can be shown that solving for a^*_2 in the aforementioned system of nonlinear equations is equivalent to solving the following equation:

$$
\begin{aligned}
& m\left(Q^{C}(a_2),\tfrac{1}{2};\psi(C^{U})\right) m\left(Q^{r}(a_2),\tfrac{1}{2};\varsigma(w^{U})\right)\Psi(r^{U},\theta^{U},\upsilon^{U}) \\
& \quad -\lambda_{DB}\, m\left(Q^{C}(a_2),\tfrac{1}{2};\psi(C^{L})\right) m\left(Q^{r}(a_2),\tfrac{1}{2};\varsigma(w^{L})\right)\Psi(r^{L},\theta^{L},\upsilon^{L}) \\
& \quad -\left(\rho^{U}-\lambda_{DB}\,\rho^{U}\right) m\left(Q^{C}(a_2),\tfrac{1}{2};\psi(C_0)\right) m\left(Q^{r}(a_2),\tfrac{1}{2};\varsigma(w_0)\right)\Psi(r_0,\theta_0,\upsilon_0)\Big] \equiv 0
\end{aligned}
\tag{8B}
$$

where,

$$
\lambda_{DB}=\frac{\left(\rho^{U}\,\kappa_{V}\,B+\kappa_{F}\right)}{\left(\rho^{L}\,\kappa_{V}\,B+\kappa_{F}+g\,B+G\right)}\,e^{\,i(C^{U-}C^{L})\omega+i\,\mathbf{\Omega}(\boldsymbol{R}^{L}-\boldsymbol{R}^{L})}
$$

and;

$$
\Psi(r,\theta,\upsilon)=M\left(Q^{\theta},S^{\theta};\varsigma^{\theta}\theta\right)M\left(Q^{\upsilon},S^{\upsilon};\varsigma^{\upsilon}\upsilon\right)exp\left\{\left(1-\tau_{p}\right)\left[\left(r-\theta\right)\upsilon^{-\frac{1}{2}}+\lambda_{r}\upsilon\right]+\varphi^{\theta}\theta+\varphi^{\upsilon}\upsilon\right\}
$$

.∎

B.2.2. Debt's Coefficients:

Like the one-factor model the coefficient of debt's value can also be determined by solving the following system of non-linear equations:

$$
\begin{aligned}
\hat{G}(\omega,\mathbf{\Omega};C^{U},\boldsymbol{R}^{U};\boldsymbol{X}) &= \left(1+\beta\right)B\,e^{\,iC^{U}\omega+i\,\mathbf{\Omega}\boldsymbol{R}^{U}} \\
\hat{G}(\omega,\mathbf{\Omega};C^{L},\boldsymbol{R}^{L};\boldsymbol{X}) &= \left((1-g)\,B-G\right)e^{\,iC^{U}\omega+i\,\mathbf{\Omega}\boldsymbol{R}^{L}}
\end{aligned}
\tag{9B}
$$

where $\boldsymbol{R^*} \equiv (w,\theta,\upsilon)^{\mathsf{T}}$, $\boldsymbol{R^*}^{U} \equiv (w^{U},\theta^{U},\upsilon^{U})^{\mathsf{T}}$, $\boldsymbol{R^*}^{L} \equiv (w^{L},\theta^{L},\upsilon^{L})^{\mathsf{T}}$, and $\mathbf{\Omega} \equiv (\psi,\varphi,\eta)$. Substituting for C and $\boldsymbol{R^*}$ in appropriate equations, $\boldsymbol{a^*} \equiv (a^*_1, a^*_2)$ is a solution for the following nonlinear system of equations:

$$
\begin{aligned}
& a^*_1(\omega,\mathbf{\Omega})\Big\{M\left(Q^{C}(a^*_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\psi(C^{U})\right)M\left(Q^{w}(a^*_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\varsigma(w^{U})\right) \\
& \qquad \times M\left(Q^{\theta},S^{\theta};\varsigma^{\theta}\theta^{U}\right)M\left(Q^{\upsilon},S^{\upsilon};\varsigma^{\upsilon}\upsilon^{U}\right) \\
& \qquad \times \exp\left\{\left(1-\tau_{p}\right)\left[\left(r^{U}-\theta^{U}\right)\upsilon^{U^{-\frac{1}{2}}}+\lambda_{r}\upsilon^{U}\right]+\varphi^{\theta}\theta^{U}+\varphi^{\upsilon}\upsilon^{U}\right\} \\
& = \left(1+\beta\right)B\,e^{\,iC^{U}\omega+i\,\mathbf{\Omega}\boldsymbol{R}^{U}}
\end{aligned}
$$

$$\begin{aligned} a*_1(\omega,\mathbf{\Omega})&\Big\{M\Big(Q^C(a*_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\psi(C^L)\Big)M\Big(Q^w(a*_2(\omega,\mathbf{\Omega})),\tfrac{1}{2};\varsigma(w^L)\Big)\\ &\times M\big(Q^\theta,S^\theta;\varsigma^\theta\theta^L\big)M\big(Q^\upsilon,S^\upsilon;\varsigma^\upsilon\upsilon^L\big)\\ &\times\exp\Big\{\big(1-\tau_p\big)\Big[\big(r^L-\theta^L\big)\upsilon^{L^{-\frac{1}{2}}}+\lambda_r\upsilon^L\Big]+\varphi^\theta\theta^L+\varphi^\upsilon\upsilon^L\Big\}\\ =\big((1-g)\,B-G\big)&e^{\,iC^L\omega+i\,\mathbf{\Omega}\boldsymbol{R}^L}\end{aligned}$$

Similar to the net benefit's case, it can be shown that solving for a_2 in the aforementioned system of nonlinear equations is equivalent to solving the following equation:

$$\begin{aligned} &m\big(Q^C(a_2),\tfrac{1}{2};\psi(C^U)\big)m\big(Q^r(a_2),\tfrac{1}{2};\varsigma(w^U)\big)\Psi(r^U,\theta^U,\upsilon^U)\\ &\quad-\lambda_{DB}\,m\big(Q^C(a_2),\tfrac{1}{2};\psi(C^L)\big)m\big(Q^r(a_2),\tfrac{1}{2};\varsigma(w^L)\big)\Psi(r^L,\theta^L,\upsilon^L)\equiv 0 \end{aligned} \tag{10B}$$

where,

$$\lambda_{DB}=\frac{\big(1+\beta\big)B}{\big((1-g)\,B-G\big)}e^{\,i(C^U-C^L)\omega+i\,\mathbf{\Omega}(\boldsymbol{R}^L-\boldsymbol{R}^L)}$$

and;

$$\Psi(r,\theta,\upsilon)=M\big(Q^\theta,S^\theta;\varsigma^\theta\theta\big)M\big(Q^\upsilon,S^\upsilon;\varsigma^\upsilon\upsilon\big)exp\Big\{\big(1-\tau_p\big)\Big[(r-\theta)\upsilon^{-\frac{1}{2}}+\lambda_r\upsilon\Big]+\varphi^\theta\theta+\varphi^\upsilon\upsilon\Big\}$$

.∎

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